19 Applications of Financial Pricing Models in Property-liability Insurance

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Abstract

This chapter provides a comprehensive survey of the literature on the financial pricing of property-liability insurance and provides some extensions of the existing literature. Financial prices for insurance reflect equilibrium relationships between risk and return or, minimally, avoid the creation of arbitrage opportunities. We discuss insurance pricing models based on the capital asset pricing model, the intertemporal capital asset pricing model, arbitrage pricing theory, and option pricing theory. Discrete time discounted cash flow models based on the net present value and internal rate of return approaches are also discussed as well as pricing models insurance derivatives such as catastrophic risk call spreads and bonds. We provide a number of suggestions for future research.

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19.1 INTRODUCTION

This chapter surveys the literature on the financial pricing of insurance and provides some extensions of the existing literature. Financial pricing differs from traditional actuarial pricing by taking into account the role played by markets in determining the price of insurance. Thus, policy prices should reflect equilibrium relationships between risk and return or, minimally, avoid the creation of arbitrage opportunities. By contrast, traditional actuarial models, such as the actuarial premium principle models (Goovaerts, de Vylder, and Haezendonck 1984), take a supply-side perspective, incorporating the assumption that prices are primarily determined by the insurer. The traditional supply-side approach is gradually being replaced by the financial pricing approach, reflecting models developed by both actuaries and financial economists.[1]

Financial theory views the insurance firm as a levered corporation with debt and equity capital. The insurer raises debt capital by issuing insurance contracts, which are roughly analogous to the bonds issued by non-financial corporations. However, insurance liabilities are not like conventional bonds but more like structured securities, where payoffs are triggered by various contingencies. The payment times and amounts for property-liability insurance policies are stochastic, determined by contingent events such as fires, earthquakes, and liability judgments. The types of risks incorporated in insurance liabilities drive both the pricing and capital structure decisions of insurers. Insurance policies also differ from bonds issued by non-financial corporations because the holders of the insurer's debt instruments are also its customers (Merton and Perold 1993). Consequently, the insurer's debt instruments should be priced to earn a fair economic profit reflecting the risks borne by the insurer. The derivation of the fair profit is one of the principal themes of this chapter.

Insurance financial pricing models have been developed to price this special class of liabilities using various strands of financial theory. The earliest models were based on the capital asset pricing model (CAPM). These models provide important insights but are too simple to be used in realistic situations, especially in light of financial research showing that factors other than the CAPM beta determine security returns (e.g., Fama and French 1993, 1996, Cochrane 1999). More promising are discrete and continuous time discounted cash flow (DCF) models, analogous to the net present value (NPV) and internal rate of return (IRR) models used in corporate capital budgeting. Option models also provide important insights into insurance pricing. The most recent research focuses on the pricing of financial instruments based on losses from property catastrophes such as hurricanes and earthquakes.

Although the primary focus in this chapter is on the theory of insurance pricing, we also briefly discuss some significant empirical contributions on the topic. We first provide a conceptual overview of the capital structure of insurance firms, with insurance policies viewed as risky debt capital. We then turn to a discussion of financial pricing models, beginning with the most basic model, the insurance capital asset pricing model (CAPM). More complex and realistic models are then discussed, including the newly-developed catastrophic risk (CAT) bonds and options.

[1] Recent actuarial papers reflecting the financial approach include Gerber and Landry (1997) and Gerber and Shiu (1998).

19.2 INSURANCE AS RISKY DEBT

Insurance companies are levered corporations that raise debt capital by issuing a specific type of financial instrument—the insurance policy. This section outlines the insurance pricing problem, describes the characteristics of insurance debt that should be reflected in financial pricing models, and discusses insurer capital structure.

In order to operate an insurance enterprise, the firm's owners commit equity capital to the firm (the reasons for doing so are discussed below) and then issue insurance policies, which are characterized by an initial premium payment, i.e., a cash inflow to the insurer, followed by a stream of cash outflows representing loss payments. During the period between the premium payment and the final satisfaction of all claims against the policy, the insurer invests the unexpended premium balance as well as the equity capital committed to the firm, receiving investment income. The equity capital is assumed to flow back to the owners as the loss obligations are satisfied. The firm's underwriting profit (the difference between premium inflows and loss outflows) and investment income expose the insurer to income tax liabilities which generate additional cash flows. Thus, the principal cash flows that must be taken into account in insurance pricing consist of premiums, losses, investment income, equity capital, and taxes.

Timing differences between the funds that flow into the company as the result of the commitment of capital and issuance of insurance policies generate the firm's assets as well as two liability accounts—the unearned premium reserve and the loss reserve. The unearned premium reserve reflects premiums that have been paid to the company for coverage not yet provided and is similar to a short-term loan (most policy coverage periods are a year or less) with no unusual risk characteristics. The loss reserve, which arises because claim payments lag premium payments and loss occurrences, represents the company's estimate of the losses it will eventually have to pay less the payments that have already been made.[2] The loss reserve is similar to an exotic option or structured security. Neither the magnitude nor the timing of loss payments are known in advance but rather depend upon contingent events such as the occurrence of accidents and the outcomes of liability lawsuits. In addition, loss cash flows can be generated by events that were unknown and/or impossible to predict when the policies were issued such as liabilities arising from exposure to environmental and asbestos exposures.[3] Because the realizations of the loss cash flows may be correlated

[2] In economic terms, the true value of the reserve is its market value which reflects the timing of expected loss payments on claims known to the insurer, an expectation of payments on accidents incurred which have not been reported to the insurer as of the statement date, a risk premium, and its value as a tax shield. However, in most industrialized countries, regulators require that insurers state their policy obligations at nominal (non-discounted) values.

[3] Actuaries often refer to the uncertainty regarding the ultimate amount of loss as *process risk*. The risks associated with the inability to accurately model the frequency and severity of all future loss events is known as *parameter risk*.

with movements in the overall financial market, insurance prices should incorporate risk premia to compensate insurers for bearing market risk.

Equity capital is the other major component of the capital structure of an insurance company. Although holding equity capital is costly due to regulation, the double taxation of dividends, and the various agency costs associated with operating an insurance company, insurers maintain capital in excess of regulatory requirements for a variety of reasons. Avoiding financial distress costs provides one important motivation for insurers to hold capital. Financial distress costs include direct costs resulting from bankruptcy as well as indirect costs which may affect the firm's ability to retain its relationships with key employees, customers, or suppliers. Merton and Perold (1993) argue that insurers also hold capital because the customers of insurers, who purchase insurance to reduce their exposure to unfavorable contingencies, are particularly concerned about the ability of the insurer to satisfy its financial obligations. Insurers also may hold equity because they issue illiquid contracts containing private information (D'Arcy and Doherty 1990, Cummins, Phillips, and Smith 1998, 2000). The benefits of this private information are only realized over time and the contracts cannot be liquidated for their full value should the firm suffer a shock to its capital resources. Finally, various agency costs, borne by the shareholders of the firm, also can be mitigated by holding additional levels of capital (e.g., Myers and Majluf 1984). Evidence that P/L insurers have strong motivations for holding equity capital is provided by the capital-to-asset ratio in the U.S. P/L industry, which equaled 33 percent in 1995. By comparison, the capital-to-asset ratio for life insurers and commercial banks are much lower, approximately 6.5 and 3.5 percent in 1995.[4]

Insurers invest primarily in financial assets, with a heavy emphasis on stocks and bonds. Insurers select assets with the objective of maximizing return while maintaining acceptable levels of credit risk exposure in their bond portfolios, exposure to price volatility from their stock portfolios, and exposure to price and exchange rate volatility from assets denominated in foreign currencies. In addition, insurers manage the duration and convexity of their asset and liability portfolios to reduce their exposure to interest rate risk (Staking and Babbel 1995). Many insurers also use off-balance-sheet contracts such as financial derivatives to manage their exposure to these same risks (Cummins, Phillips, and Smith 1997, 1998, Santomero and Babbel 1997).

The risks that should be taken into account in pricing insurance contracts are summarized in Table 1. Insurance pricing models differ in the degree to which these risks are recognized. The existing insurance financial pricing models tend to focus on systematic risk, inflation risk, and interest rate risk. More research is needed on unified models that incorporate all types of risk.

[4] The capital-to-asset ratios are from the Federal Reserve Flow of Funds Accounts (Washington, D.C.: Board of Governors of the Federal Reserve System).

Table 1
Pricing Characteristics and Risks in Property-Liability Insurance

Uncertainty Regarding Frequency and Severity (Process Risk)
Uncertainty Regarding Models to Estimate Losses (Parameter Risk)
Interest Rate (Duration and convexity) Risk
Inflation Risk
Payout Pattern Risk
Systematic (Market) Risk
Default Risk

19.3 A SIMPLE CAPM FOR INSURANCE PRICING

The first financial models of the insurance firm were based on a very simple algebraic approach. The first model of this type was developed by Ferrari (1969). His paper presents the basic algebraic model of the insurer but does not link the model to the concept of market equilibrium. An important advance in insurance financial pricing was the linkage of the algebraic model of the insurance firm with the capital asset pricing model (CAPM). The resulting model is often called the *insurance CAPM.*

The insurance CAPM was developed in Cooper (1974), Biger and Kahane (1978), Fairley (1979), and Hill (1979). The derivation begins with the following simple model of the insurance firm:

$$\tilde{Y} = \tilde{I} + \tilde{\Pi}_u = \tilde{r}_a A + \tilde{r}_u P \tag{1}$$

where $\tilde{Y}, \tilde{I}$ = net income and investment income, respectively,
$\tilde{\Pi}_u$ = underwriting profit (loss) = premium income less expenses and losses,
A = invested assets of the firm
P = premiums collected from policyholders to compensate insurers for the risks they underwrite,
$\tilde{r}_a$ = rate of investment return on assets, and
$\tilde{r}_u$ = rate of return on underwriting (as a proportion of premiums).

Tildes indicate stochastic variables. Writing (1) as return on equity and using the balance sheet identity $A = R + G$, where R = (undiscounted) reserves and G = equity, one obtains:

$$\tilde{r}_e = \tilde{r}_a\left(\frac{R}{G}+1\right)+\tilde{r}_u\frac{P}{G} = \tilde{r}_a(ks+1)+\tilde{r}_u s \tag{2}$$

where s = P/G = the premiums-to-equity (or premiums-to-surplus) ratio, and
k = R/P = the liabilities-to-premiums ratio (*funds generating factor*).

Equation (2) indicates that the rate of return on equity for an insurer is generated by both *financial leverage* ($R/G + 1$) and *insurance leverage* (P/G). The leverage factor for investment income is a function of the premium-to-surplus ratio and the funds generating factor. The latter approximates the average time between the policy issue and claims payment dates. The underwriting return is leveraged by the premium-to-surplus ratio. Taking expectations in (2), one obtains the insurer's expected return on equity.

Equation (2) is essentially an accounting model. The model is given economic content by assuming that the equilibrium expected return on the insurer's equity is determined by the CAPM. The CAPM formula for the expected return on the insurer's stock is

$$E(\tilde{r}_e) = r_f + \beta_e[E(\tilde{r}_m) - r_f] \tag{3}$$

where $E(\tilde{r}_e)$ = expected return on the insurer's equity capital,
β_e = the insurer's equity beta coefficient = $\text{Cov}(\tilde{r}_e, \tilde{r}_m)/\text{Var}(\tilde{r}_m)$,
$E(\tilde{r}_m)$ = expected return on the market portfolio, and
r_f = the risk-free rate of interest.

The insurance CAPM is obtained by equating the CAPM rate of return on the insurer's equity with the expected return given by equation (2) and solving for the expected underwriting profit.[5] The result is:

$$E(\tilde{r}_u) = -k\, r_f + \beta_u[E(\tilde{r}_m) - r_f] \tag{4}$$

where $\beta_u = \text{Cov}(\tilde{r}_u, \tilde{r}_m)/\text{Var}(\tilde{r}_m)$ = the beta of underwriting profits.

The insurer must earn (in expectation) the return $E(\tilde{r}_u)$ on underwriting in order to avoid penalizing equity (if the return is too low) or charging policyholders too much (if the return is too high). The first term in equation (4), $-k\, r_f$, represents an interest credit for the use of policyholder funds. The second component of $E(\tilde{r}_u)$ is the insurer's reward for risk-bearing: the underwriting beta multiplied by the market risk premium. The risk premium reflects only systematic risk, i.e., policies are treated as free of default risk.

Several limitations of the insurance CAPM have motivated researchers to seek more realistic models. One problem is the use of the funds generating factor (k) to represent the payout tail. Myers and Cohn (1987) argue that k is only an approximation of the discounted cash flow (DCF) approach. A second limitation is that the model ignores default risk. As a practical matter, errors in estimating underwriting betas

[5] The derivation also uses the CAPM pricing relationship for the insurer's expected asset return, $E(\tilde{r}_a)$, i.e., $E(\tilde{r}_a) = r_f + \beta_a[E(\tilde{r}_m) - r_f]$ as well as the relationship $\beta_e = \beta_a(ks + 1) + \beta_u s$.

can be significant (Cummins and Harrington 1985). A final serious limitation is that most recent studies have shown that security returns are related to other factors in addition to the CAPM beta coefficient. A more modern version of the insurance CAPM could easily be developed that incorporate multi-factor asset pricing models. Most of the models discussed below deal with one or more of the limitations of the CAPM.

19.4 DISCRETE TIME DISCOUNTED CASH FLOW (DCF) MODELS

Paralleling corporate finance, DCF models for insurance pricing have been developed based on the net present value (NPV) and the internal rate of return (IRR) approaches. The NPV approach was applied originally by Myers and Cohn (1987) and extended by Cummins (1990) and Taylor (1994). The NPV model is an application of *adjusted present value* (APV) method, which requires each cash flow to be discounted at its own risk-adjusted discount rate (RADR) (see Brealey and Myers 1996). The IRR approach was originally developed by the National Council on Compensation Insurance (NCCI) and is further discussed in Taylor (1994).[6] In this section we provide a general discussion of the DCF approach to insurance pricing using notation taken from Taylor (1994).[7] Taylor's model is more rigorously developed than earlier models, and he explicitly derives the conditions under which the NPV and IRR models give the same results.

We begin by defining some additional notation. Specifically, let

P = the premium paid by policyholders for insurance coverage,
a_t = the proportion of the premium paid at time t,
$\tilde{L}$ = the total amount of losses under the policy,
c_t = the proportion of losses paid at time t,
$\tilde{L}_t = c_t\tilde{L}$ = the amount of the loss payment at time t,
$E(\tilde{r}_l) = r_f + \beta_l[E(\tilde{r}_m) - r_f]$ = the expected value of the loss discount rate $\tilde{r}_l$,
$E(\tilde{r}_a) = r_f + \beta_a[E(\tilde{r}_m) - r_f]$ = the expected return on the insurer's invested assets,
$\beta_x = \text{Cov}(\tilde{r}_x, \tilde{r}_m)/\text{Var}(\tilde{r}_m)$ = the beta coefficient for cash flow x (x = l, a),
G_t = the insurer's equity capital at time t, and
τ = tax rate for investment and underwriting income.[8]

[6] Of course, the internal rate of return model in insurance is subject to the same well-known pitfalls that have been identified in corporate finance more generally. See, for example, Brealy and Myers (1996). However, as Brealy and Myers point out, "used properly, it gives the same answer" as the net present value (NPV) method (Brealy and Myers, p. 85).

[7] Taylor (1994) provides the set of conditions under which the net present value model and the internal rate of return models will yield identical premia. Our discussion is a simplified version of his model. The reader is referred to the original paper for more details.

[8] Although we believe that our modeling of income taxes is reasonably generic, the models would have to be modified for use in jurisdictions that have other types of tax formulas.

As above, tildes indicate random variables. The insurer is assumed to issue policies at time 0. In general, premiums are received at times $\{0, 1, \ldots, T-1\}$ and losses are paid at times $\{1, 2, \ldots, T\}$, where T is the last loss payment date. We assume that $c_t > 0$ at all times $\{1, 2, \ldots, T\}$, but premium payments may be zero at some possible premium payment dates. An important special case is where all premiums are paid at time zero, i.e., $a_t = 0$, $t \neq 0$. Expenses (other than loss payments) are assumed to be zero. The asset and liability discount rates, the risk-free rate, the expected return on the market portfolio, and the beta coefficients are all assumed to be constant over the payout period. Insurer underwriting profits and investment income are taxed at the constant rate τ.

An important feature of the discounted cash flow approach to insurance pricing is the concept of the *surplus flow*. The insurer is assumed to commit equity capital (surplus) to the enterprise at time 0, and the capital is assumed to flow back to the insurer over the loss payment period. A specific pattern of surplus flow is required in order for the net present value and IRR methods to yield identical premiums. Myers and Cohn assumed that surplus is released as losses are paid. However, Taylor (1994) shows that their assumption will not lead to equivalency of the NPV and IRR premiums. Taylor shows instead that the surplus must be released in proportion to reductions in reserves. We return to this point below.

The insurer's market value balance sheet consists of the market value of its assets on one side and the market value of its debt and equity on the other. Debt capital consists of loss reserves, i.e., no bonds or other types of non-insurance debt capital are used. The market value of liabilities can be defined as:

$$R_t^m = \sum_{t=1}^{T}\left[\frac{L_t(1+v_t)}{[1+E(r_l)]^t} - \frac{P_{t-1}}{[1+r_f]^{t-1}}\right] = R_t^{ml} - R_t^{mp} \tag{5}$$

where R_t^m = the market value of reserves at time t,

v_t = loading factor applied to expected costs of period t in their contribution to loss reserves, and

R_t^{ml}, R_t^{mp} = the loss and premium components, respectively, of equation (5).

The factor v_t reflects loadings that are held in reserves until realization at time t. The loadings are needed to pay the taxes on underwriting and investment income (see below). The premium component R_t^{mp} represents a receivables account and could be equivalently treated as an asset item. The *leverage factor* can be defined as $\theta_t^{-1} = R_t^{ml}/G_t$, $t = 0, 1, \ldots, T$.

Taylor (1994) derives the following formula for the cash flows to/from the insurer's owners:

$$\tilde{F}_t = (G_{t-1} - G_t) + (1-\tau)\tilde{\Pi}_t \tag{6}$$

where $\tilde{F}_t$ = net cash flow to (from) owners in period t. $\tilde{\Pi}_t$ = the insurer's profit at time t, defined as follows:[9]

$$\tilde{\Pi}_t = G_{t-1}(1-\tau)\left[E(\tilde{r}_a)+[E(\tilde{r}_a)-E(\tilde{r}_l)]\frac{R^{ml}_{t-1}}{G_{t-1}}+v_t\frac{L_t}{G_{t-1}}-[E(\tilde{r}_a)-r_f]\frac{(R^{mp}_{t-1}-P_{t-1})}{G_{t-1}}+\tilde{z}_{t-1}\right] \quad (7)$$

The profit is after-tax and has five components, corresponding to the five terms inside the brackets in equation (7). The first, $E(\tilde{r}_a)$, corresponds to the investment income earned on the insurer's equity. The second, equal to $[E(\tilde{r}_a) - E(\tilde{r}_l)]$ leveraged by $(\theta_{t-1})^{-1}$, reflects investment income on reserves less the rate of return credit needed to write up discounted reserves for an additional period as they approach maturity, the latter being a deduction in determining taxable income (Cummins 1990). The third, involving v_t, represents the release to profits of the loading margin in the loss reserve. The fourth term represents the reduction in income attributable to premiums not yet received by the insurer; and the fifth term, $\tilde{z}_{t-1}$, is a mean-zero random variable to capture deviations of losses and investment income from their expected values.

The insurer's return on equity (ROE) in period t can be obtained by dividing through equation (6) by G_{t-1}. Because we have assumed no changes in the underlying variables such as expected investment returns and taxes, the expected return on equity should be constant over the entire runoff period. Thus, it is of interest to inquire about the conditions that will lead to a constant ROE. Using equations (6) and (7), the CAPM formulas for $E(\tilde{r}_a)$ and $E(\tilde{r}_l)$, and the definition of θ_t, ROE can be written as

$$\tilde{r}_e = (1-\tau)\left[E(\tilde{r}_a)+[E(\tilde{r}_a)-E(\tilde{r}_l)]\theta_{t-1}^{-1}\left(1-\frac{\beta_a}{\beta_a-\beta_l}\frac{R^{mp}_{t-1}-P_{t-1}}{R^{ml}_{t-1}}\right)+v_t\frac{L_t}{G_{t-1}}+\tilde{z}_t\right] \quad (8)$$

Considering the second term inside of the brackets, involving $(\theta_{t-1})^{-1}$, it is clear that this term will be constant if $\theta_{t-1} = \theta$, $\forall\ t$, where θ is a constant, and the term involving R^{mp}_t and R^{ml}_t is constant. An important special case where this will occur is when all premiums are received at time zero. Otherwise, the condition imposes a constraint on the ratio of the reserve for deferred premiums to the reserve for unpaid losses.

In the no-tax case where $v_t = 0\ \forall\ t$, the v_t term in equation (8) vanishes, so we do not need to worry about this term creating non-constant ROE. When $v_t \neq 0$, the condition that the v_t term in (8) must satisfy in order for ROE to be constant is the following:

$$v_t\frac{L_t}{R^{ml}_{t-1}} = \frac{\theta\tau}{1-\tau}ROE \quad (9)$$

[9] The profit is the amount that must be earned in order for the insurer to earn its cost of capital on the policy. We are not suggesting that monopoly rents play a role in this model.

where ROE = the constant value for ROE being sought in the analysis. Because the right hand side of equation (9) is a constant, this condition implies that the emergence of profits should vary inversely with the ratio L_t/R^{ml}_{t-1}, i.e., profit emerges proportionately to the ratio of paid losses to the (present valued) loss reserve. This differs significantly from the approach in Myers and Cohn (1987), who assume that "underwriting profits are accrued as losses are *paid*" (p. 68, emphasis added). Taylor's result in equation (9) shows that the MC assumption about the emergence of profit will not lead to ROE being constant over time, even when the underlying parameters are constant.

We are now ready to compare the IRR and NPV models. Consider first the IRR model. This model specifies that the premium P is the solution of the following equation:

$$\sum_{t=0}^{T} \frac{E(\tilde{F}_t)}{[1+E(\tilde{r}_e)]^t} = 0 \tag{10}$$

where $\tilde{F}_t$ is given by equation (6). It is convenient in discussing the method to assume that all premiums are paid at time zero. Then equations (8) and (9) imply that

$$E(\tilde{r}_e) = E(\tilde{r}_a) + [E(\tilde{r}_a) - E(\tilde{r}_l)]\theta^{-1} - \tau[E(\tilde{r}_a) + [E(\tilde{r}_a) - E(\tilde{r}_l)]\theta^{-1}] + (1-\tau)v_t \frac{L_t}{G_{t-1}} \tag{11}$$

But by equation (10) the last two terms in (11) sum to zero, so $E(\tilde{r}_e) = E(\tilde{r}_a) + [E(\tilde{r}_a) - E(\tilde{r}_l)]\theta^{-1}$, where θ^{-1} is the constant ratio of the present value of unpaid losses to capital. Therefore, under these conditions we have a constant ROE.

This result has several important implications. First, the constant ROE generated by the model is the required rate of return implied by the CAPM, which can be rewritten as $E(\tilde{r}_e) = E(\tilde{r}_a)(1 + R^{ml}/G) - E(\tilde{r}_l)\ (R^{ml}/G)$, where $R^{ml}/G = \theta^{-1}$ = the ratio of the present value of reserves to equity capital, which is constant for all t. Thus, the insurer earns a leveraged return on assets at rate $E(\tilde{r}_a)$ and pays for the use of policyholder funds at the rate $E(\tilde{r}_l)$, where both $E(\tilde{r}_a)$ and $E(\tilde{r}_l)$ are determined by the CAPM. Second, the profit loadings (v_t) emerge at exactly the time and amount needed to offset the income tax on the ROE, so that the insurer earns the pre-tax ROE. The policyholder pays the firm's income tax in accordance with the argument that the owners will not commit capital to the insurer if it is subjected to another layer of taxation because the owners have the option of investing directly in the capital markets. And, third, like the insurance CAPM, the model does not recognize insolvency risk. Thus, θ is indeterminate, and there is nothing explicitly in the model to prevent the insurer from infinitely leveraging the firm. Thus, the model incorporates the implicit assumption that market discipline or regulation prevent infinite leveraging.

The fourth implication provides a link between the IRR and the MC net present value models. This argument is a bit more subtle and the reader is referred to Taylor (1994) for a rigorous proof. However, the intuition is that the equation defining the IRR (equation (10)) implies that no profit emerges at time zero. This in turn implies that $R_0^m = 0$ in equation (5) and thus that the premium satisfies the following equation:

$$\sum_{t=1}^{T} \frac{P_{t-1}}{(1+r_f)^{t-1}} = \sum_{t=1}^{T} \frac{L_t(1+v_t)}{(1+E(\tilde{r}_l))^t} \tag{12}$$

If v_t is zero for all t, equation (12) is exactly the Myers-Cohn model for the case of no taxes.

To produce a constant CAPM return on equity, one could calculate premiums using the IRR model in equation (12). Alternatively, we can restate the MC model using the surplus release pattern postulated by Taylor. To do this, we first note that the APV approach requires that the insurer's tax liability be broken down into its components, with each component discounted at the appropriate rate. The insurer's expected tax liability can be disaggregated as follows:

$$Tax_t = \tau\{E(\tilde{r}_a)[G_{t-1} + R_{t-1}^{ml} - (R_{t-1}^{mp} - P_{t-1})] - E(\tilde{r}_l)R_{t-1}^{ml} + r_f(R_{t-1}^{mp} - P_{t-1})\} \tag{13}$$

The first expression in equation (13), equal to the expected investment return times the bracketed expression, is the investment return on the insurer's assets at the start of the period, where assets (A_{t-1}) equal equity (G_{t-1}) plus reserves ($R^{ml} - R^{mp}$) plus premiums received (P_{t-1}). The second term ($-E(\tilde{r}_l)\ R_{t-1}^{ml}$) is a deduction for losses paid and for the write-up of the remaining loss reserve to reflect the reduced time to maturity. The third component is the interest write up to accrue the premium account towards maturity, i.e., a financing charge for unpaid premiums.

The present value of the tax components is added to the present value of losses to obtain the NPV premium. To obtain the premium, it is necessary to specify a RADR for each component of the tax. For the last term, the answer is obvious: $r_f(R_{t-1}^{mp} - P_{t-1})$ is a riskless flow and therefore is discounted at the risk free rate, r_f. For the first term, which is a risky investment flow, determined by the risky rate of return $\tilde{r}_a$, the answer is not so obvious. However, it turns out that this flow as well is discounted at the risk free rate. This result is known as the *Myers' theorem*, developed by Myers (1984) and proved more rigorously in Derrig (1994) and Taylor (1994).

The Myers' theorem can be demonstrated easily. Assume an investment of 1 at time 0 in the risky asset. The return on the asset, which is unknown at time 0, will be $\tilde{r}_a$. The investor will receive this risky return at time 1 and will pay a tax of $\tau\ \tilde{r}_a$. The question is: what is the present value at time 0 of this tax flow? The result is obtained by observing that the investor is able to deduct the amount of the initial investment (i.e., 1) before paying the tax. Although the investment return is risky, the deduction is not.

We are seeking the following present value:

$$\text{PV(Tax)} = \text{PV}(\tau\tilde{r}_a) = \tau\text{PV}(\tilde{r}_a)$$

However, we also recognize that the investor will have both the principal and interest at time 1, i.e., $(1 + \tilde{r}_a)$, and can deduct 1 before paying the tax. Therefore, we can write

$$\mathrm{PV(Tax)} = \tau \mathrm{PV}[(1+\tilde{r}_a)-1]$$

But we have assumed that capital markets are efficient, so the present value of the risky amount $(1 + \tilde{r}_a)$ is 1, i.e., the appropriate discount factor for this term is $(1 + \tilde{r}_a)$. The deduction of 1 is riskless and thus is discounted at the risk free rate. Therefore, we have

$$\mathrm{PV(Tax)} = \tau \mathrm{PV}[(1+\tilde{r}_a)-1] = \tau[1-1/(1+r_f)] = \tau r_f/(1+r_f)$$

that is, the present value of the tax on a risky investment of \$1 is the risk-free rate times the tax rate, discounted at the risk-free rate.

The only component for which we still need a discount rate is the loss deduction. The loss deduction depends on risky losses so that the appropriate discount rate is $E(\tilde{r}_l)$. Consequently, the revised form of the Myers-Cohn model, which we term the *Myers-Cohn-Taylor (MCT) net present value model*, is given by:

$$\sum_{t=0}^{T-1} Pa_t = \sum_{t=1}^{T} L \frac{c_t}{[1+E(\tilde{r}_L)]^t} + \tau\left[\sum_{t=1}^{T} \frac{r_f(A_{t-1}+R_{t-1}^{ml})}{(1+r_f)^t} - \sum_{t=1}^{T} \frac{E(\tilde{r}_l)R_{t-1}^{ml}}{[1+E(\tilde{r}_L)]^t}\right] \tag{14}$$

Premiums based on this model will generate a constant (expected) cost of capital throughout the runoff period for the policies being priced, and it will give the same premium as the IRR model.

Although the discrete time models we have discussed here are useful and practical financial pricing models, they are not without limitations. E.g., multi-factor models that price various sources of risk should be used instead of the CAPM in discounting risky cash flows (see Cochrane 1999). Research identifying the sources of priced risk in insurance markets would be an important advance in this field.

Recent work in the theory of risk management also suggest that the models presented here may be in need of further development. Because of various capital market imperfections, the cost of raising capital external to the firm will be more costly than capital generated from internal sources (Froot, Scharfstein, and Stein 1993). Thus, firms have an incentive to manage risk at the individual firm level to decrease the likelihood of having to raise costly external capital. Froot and Stein (1998) have developed a capital budgeting model that allows for the possibility that external capital is more costly than internal capital. They argue that the discount rate on lines of business which co-vary positively with overall firm capital levels should have higher discount rates than lines of business which co-vary negatively. Their work suggests that in the presence of financing imperfections, the optimal discount rate will depend not only upon the economy-wide systematic risks, but will also include a firm and line specific adjustment determined by how the losses underwritten on a particular line of business are expected to correlate with the internal capital levels of the firm. Future research validating (or rejecting) this hypothesis would greatly increase our understanding about how intermediaries price their products and also about the sources of friction in insurance markets.

19.5 OPTION PRICING MODELS

Like options, insurance policies can be thought of as derivative financial assets (contingent claims) with payments that depend upon changes in the value of other assets. Payments under primary insurance policies are triggered by changes in the value of insured assets, while reinsurance payments depend upon the experience of the primary insurer. Thus, it is natural to consider option models for pricing insurance.

The basic paradigm for pricing derivatives is the no-arbitrage principle. No arbitrage exists in perfect and frictionless markets if the payoffs on the derivative security can be replicated using existing securities with known prices. The price of the derivative is found by forming a portfolio of primitive securities whose payoffs exactly replicate the payoffs on the derivative. Since the prices of the primitive securities are assumed to be known, the price of the derivative must be exactly equal to the value of the replicating portfolio.

Financial economics theory has shown that when markets are complete and arbitrage free there exists a pseudo-probability measure, known as the *risk-neutral measure*, under which *all* uncertain cash flow streams can be priced using the risk-free rate of interest (Duffie 1996). The equal return feature is just a fiction, of course—returns on most assets, including options, are not actually equal to the risk-free rate. Rather, the risk neutral valuation technique prices securities *as if* returns were risk-free. Thus, the price of any uncertain cash flow stream can be determined by taking expectations of the future cash flows using the risk neutralized probability distribution and then discounting at the risk-free.

The discussion of option pricing models of insurance we present in this chapter parallels the evolution of the literature in which the principles of no arbitrage and risk-neutral valuation are standard assumptions. However, it should be noted that the assumptions of no-arbitrage and completeness in insurance markets are non-trivial as they imply there exists a sufficient number of linearly independent financial instruments to hedge all risks and replicate the payoffs on any insurance contract. This assumption is more realistic for some insurance products than for others. For example, the valuation of crop insurance using no-arbitrage arguments is relatively straightforward since the underlying risk (the commodity price) can be replicated using the spot markets and existing traded securities such as futures and options. Identifying the set of securities which completes the market for other insurance products is more difficult and suggests a possible limitation of this literature as well as offering opportunities for future research. For example, the literature on incomplete markets has received little attention in the context of insurance financial pricing.

19.5.1 Basic Option Models In Insurance

Single period option models provide some important insights into insurance pricing.[10] A simple example is the pricing of excess of loss reinsurance on a portfolio of primary insurance policies which is sufficiently large and has loss severity sufficiently small so that the evolution of claim costs can be approximated by a Brownian motion process. Consider such an excess of loss reinsurance agreement in which the reinsurer agrees to pay the losses of the primary insurer in the event these losses exceed a fixed retention amount M up to a maximum limit of U. In this case, the insurance policy is a call option spread, paying $\{\text{Max}[0, Y - M] - \text{Max}[0, Y - U]\}$ at maturity, where Y = losses. Under the appropriate conditions, the Black-Scholes approach leads to the following formula for the reinsurance premium:

$$P_R = e^{-r_f\tau} \int_M^U (X - M) \frac{1}{X\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln X - \mu}{\sigma}\right)^2} dX + [M - U]\left[1 - N\left(\frac{\ln U - \mu}{\sigma}\right)\right] \tag{15}$$

where $\mu = r_f - \sigma^2/2$ and $N(\cdot)$ is the standard normal distribution function.

Another application of option modeling in insurance is to analyze insolvency risk. This application utilizes the put-call parity formula:

$$A = C(A, L, \tau) + [Le^{-r_f\tau} - P(A, L, \tau)] \tag{16}$$

where A = the value of firm assets,

L = the value of firm liabilities,

$C(A, L, \tau)$ = a call option on asset A, with striking price L, and time to maturity τ, and

$P(A, L, \tau)$ = a put option on asset A, with striking price L and time to maturity τ.

The options are assumed to be European options, implying that they can only be exercised at the maturity date. The option model of the firm expresses the ownership interest as the value of the call option because the owners have the right to receive the residual value of the firm at the expiration date. If $A > L$ at that date, the owners pay off the liabilities and receive the amount $A - L$. If $A < L$, the owners default,

[10] Relatively early articles using single period option models to study insurance problems include Merton (1978), Doherty and Garven (1986) and Cummins (1988).

turning the firms assets over to the debt holders. The value of the policyholders' interest in the firm (i.e., the fair value of the insurance at any time prior to the option exercise data) is given by the bracketed expression in (16), the riskless present value of liabilities minus the put value. The put represents a discount in the price of insurance to reflect the expected value of the owners' option to default if $A < L$ and is called the *insolvency put.*[11] Thus, the fair price of insurance is the riskless present value of losses less the insolvency put.

Basic option models have some limitations that restrict their applicability to many real world insurance problems.[12] Three examples are: (1) The models are restricted to a single payoff, even though most real-world property-liability policies have multiple cash flows. (2) There is only one class of liabilities, whereas most insurers write multiple lines of insurance. And (3) they require that the optioned variable be continuous. Thus, discrete jumps in loss values are ruled out. To relax the multiple period assumption, it would be possible to adapt other types of financial models such as the *compound options model* discussed in Geske (1977, 1979) or perhaps a coupon bond model. In the following sections, we discuss some attempts to generalize the models to incorporate multiple classes of liabilities and jump processes.

19.5.2 A Multi-Class Option Model

Because most insurers are multiple-line operations, it is of interest to extend the basic insurance option model to the case of multiple liabilities (Cummins and Danzon 1997, Phillips, Cummins, and Allen 1998). To conserve notation, the model is derived with two liability classes. Assume that insurer assets and liabilities follow diffusion processes:

$$
\begin{aligned}
dA &= \mu_A A dt + \sigma_A A dz_A \\
dL_1 &= \mu_{L_1} L_1 dt + \sigma_{L_1} L_1 dz_{L_1} \qquad (17) \\
dL_2 &= \mu_{L_2} L_2 dt + \sigma_{L_2} L_2 dz_{L_2}
\end{aligned}
$$

where A, L_1, L_2 = market values of assets and liabilities (classes 1 and 2),
μ_A, σ_A = drift and diffusion parameters for assets,

[11] Cummins (1988) uses the put-call parity relationship to obtain the premium for guaranty insurance as the value of the put, $P(A, L, \tau)$.

[12] Nevertheless, the simple option models may perform better than might be expected. D'Arcy and Garven (1990) tested the performance of several financial pricing models in explaining actual underwriting profit margins over a sixty year period ending in 1985. They found that the most accurate models were basic option pricing models (Doherty and Garven 1986, Cummins 1988) and an industry rule of thumb model, the total return model. The insurance CAPM and the NPV models did not perform as well as the option and total return models.

μ_{Li}, σ_{Li} = drift and diffusion parameters for liability class i, i = 1, 2, and
dz_A, dz_{L1}, dz_{L2} = increments of the Brownian motion processes for the asset and liability classes 1 and 2.

The Brownian processes are related as follows: $dz_A\, dz_{L1} = \rho_{A1}\, dt$, $dz_A\, dz_{L2} = \rho_{A2}\, dt$, $dz_{L1}\, dz_{L2} = \rho_{12}\, dt$, where ρ_{Ai}, i = 1,2, = instantaneous correlation coefficients between the Brownian processes for assets and liability classes 1 and 2, respectively, and ρ_{12} = the instantaneous correlation coefficient for liability classes 1 and 2.

Both assets and liabilities are assumed to be priced according to an asset pricing model, such as the inter-temporal capital asset pricing model (ICAPM), implying the following return relationships:

$\mu_A = r_f + \pi_A$, for assets, and
$\mu_{Li} = r_{Li} + \pi_{Li}$, for liability classes i = 1, 2.

where r_{Li} = the inflation rate in liability class i, and
π_j = the market risk premium for asset $j = A, L_1, L_2$.

The Fisher relationship is assumed to hold so that $r_f = r + r_I$, where r = the real rate of interest and r_I = economy-wide rate of anticipated inflation. The economy-wide rate of inflation will not in general equal the inflation rates on the two classes of insurance liabilities. If assets (and liabilities) are priced according to the ICAPM, the risk premium would be:[13]

$$\pi_j = \rho_{jm}(\sigma_j/\sigma_m)[\mu_m - r_f]$$

where μ_m, σ_m = the drift and diffusion parameters of the Brownian motion process for the market portfolio, and
ρ_{jm} = the correlation coefficient between the Brownian motion process for asset j and that for the market portfolio.

The value of an option on the two-liability insurance company can be written as $P(A, L_1, L_2, \tau)$, where τ = time to expiration of the option. Differentiating P using Ito's lemma and invoking the ICAPM pricing relationships for assets and liabilities yields the following differential equation:

$$Pr_f = r_f P_A A + r_{L_1} P_{L_1} L_1 + r_{L_2} P_{L_2} L_2 - P_\tau + \frac{1}{2}\sigma_A^2 P_{AA} A^2 + \frac{1}{2}\sigma_{L_1}^2 L_1^2 P_{L_1 L_1} + \frac{1}{2}\sigma_{L_2}^2 P_{L_2 L_2} L_2^2 + P_{AL_1} A L_1 \sigma_{A1} + P_{AL_2} A L_2 \sigma_{A2} + P_{L_1 L_2}\sigma_{12} L_1 L_2 \quad (18)$$

[13] Alternatively, the risk premium could be defined using the consumption CAPM (Breedon 1979). The consumption CAPM assumes risk premia are related to the rate of return on aggregate real consumption instead of assuming that asset risk premia are related to movements in securities markets. Empirical tests of the consumption CAPM, however, suggest the model does no better explaining security returns than

Risk and the drift parameters (μ_j) have been eliminated by using the ICAPM and taking expectations. This also could be done by using a hedging argument, provided that appropriate hedging assets are available.

The next step is to use the homogeneity property of the option model to express the model in terms of the asset-to-liability ratio x, the option value-to-liability ratio $p = P/L$, and the liability proportions $w_1 = L_1/L$ and $w_2 = L_2/L$, where $x = A/L$ and $L = L_1 + L_2$. The result is the following differential equation:

$$pr_n = xp_x r_n - p_\tau + \frac{1}{2}x^2 p_{xx}\sigma_n^2 \tag{19}$$

where $r_n = r_f - w_1 r_{L1} - w_2 r_{L2}$,

$\sigma_n^2 = \sigma_A^2 + w_1^2\sigma_{L1}^2 + w_2^2\sigma_{L2}^2 - 2w_1\sigma_{A1} - 2w_2\sigma_{A2} + 2w_1 w_2\sigma_{12}$,

σ_j^2 = the diffusion parameter for process j ($j = A$ = assets, $j = L1, L2$ = liability classes 1 and 2), and

σ_{jk} = the covariance parameter for processes j and k, A = assets, 1,2 = classes 1 and 2.

Equation (19) is the standard Black-Scholes differential equation, where the optioned asset is the asset-to-liability ratio (x).

This model can be used to price various contingent claims on the insurer by solving the equation subject to the appropriate boundary conditions. For example, the call option $c(x, 1, \tau)$ = the value of owners' equity, is the solution to equation (19) with boundary condition $c(x, 1, 0) = \text{Max}(x - 1, 0)$. The put option $g(x, 1, \tau)$ = the guaranty fund premium is the solution of (19) with boundary condition $g(x, 1, 0) = \text{Max}(1 - x, 0)$. The value of policy liabilities is obtained from the parity relationship as $b(x, 1, \tau) = \exp(-r\tau) - g(x, 1, \tau)$. The striking price in each case is equal to 1 because of the normalization of asset and option values by L. The option values are given by the usual Black-Scholes call and put option formulas (see Ingersoll (1987)).[14]

19.5.3 Implications of the Multi-Class Model

A number of interesting implications about insurance markets can be gleaned from equation (19). For example, the equation reveals that a portfolio effect exists for insurers that write multiple policies or multiple lines. To be specific, assume the existence of two insurers, with assets A_i, liabilities L_i, and risk parameters σ_i, $i = 1, 2$. The put values for the two insurers separately are $g_i(A_i, L_i, \tau)$, $i = 1, 2$. Now suppose that the two companies are merged, with no change in the asset or liability parameters. Assuming there is no correlation between the asset and liability processes and the correla-

does the traditional CAPM (Breeden, Gibbons, and Litzenberger 1989). Thus, the value of using the consumption CAPM in insurance pricing is an open question.

[14] Using the homogeneity property, the options on x can be rescaled in dollars by multiplying by L.

tion coefficient between the liability processes ρ_{12} is not equal to one, it is easy to show the put value for the merged insurer, $g(A_1 + A_2, L_1 + L_2, \tau)$ must be less than or equal to the put values of the two insurers separately owing to the convexity of European puts (Merton 1973). Intuitively, the portfolio of puts from the separate insurers is worth at least as much as the put on the portfolio because situations exist where one of the individual puts finishes in the money but the portfolio does not. Thus, equation (19) implies that value is created by pooling different classes of risks in a portfolio and multiple line insurers have an advantage over mono-line insurers in that they can offer equally safe insurance with less capital as long as the liability processes are not perfectly correlated

The multi-class option model has also been used by Cummins and Danzon (1997) to gain some insights into the supply of insurance. They consider a company which has an existing portfolio of policies L_1 with one year until maturity. Its assets are A_1, and its existing portfolio will pay no additional premiums. The company has the opportunity to write a new block of policies, L_2. To write the new policies, it may have to issue new equity. The company is seeking a strategy for issuing new equity and pricing the new policies.

Assuming that markets are efficient and that the policyholders know the characteristics of insurers, pricing will depend upon the *liquidation rule*, i.e., the rule governing the disposition of the company's assets in the event of insolvency. Assume that the liquidation rule compensates policyholders in proportion to the nominal value of their claims against the company, so that policy class i obtains proportion $w_i = L_i/(L_1 + L_2)$ of assets.[15] Then, the fair premium for the new policyholders is: $L_2\,[\exp(-r\tau) - w_2 g_S(x, \tau)]$, where $g_S(x, \tau)$ = the put option on the company after the new policies and new equity are issued and $x = (A_1 + A_2)/(L_1 + L_2)$.

Because the pricing rule of the new policyholders is satisfied for a wide range of x values, the amount of the equity issue is indeterminate unless additional structure is imposed on the problem. For example, equity owners could gain by issuing the new policies and obtaining little or no new equity. This would expropriate value from the existing (class 1) policyholders without affecting the new policyholders, who pay the fair value for their coverage. In a competitive, efficient market, it is unlikely that the equity owners would be able to gain by expropriation. Expropriating value from the old policyholders would adversely affect the firm's reputation and its future cash flows. For example, the new policyholders might not be willing to pay the "fair value" if it appears that the owners have a history of expropriating wealth from policyholders by changing the capital structure or risk characteristics of the firm.

Assume that the firm's objective is for the value of equity after the equity/policy issue to be at least as large as the sum of its equity before the equity/policy issue and the amount of new capital raised (E), i.e.,

[15] This is consistent with the way insurance insolvencies are handled in practice (National Association of Insurance Commissioners 1993).

$$C_s(A_1 + A_2, L_1 + L_2, \tau) \geq C_1(A_1, L_1, \tau) + E \tag{20a}$$

Substituting for the value of the call options on the both sides of the inequality yields:

$$\begin{aligned} &A_1 + A_2 - (L_1 + L_2)e^{-r\tau} + (L_1 + L_2)p(x, \tau) \geq A_1 - L_1 e^{-r\tau} + L_1 p_1(x_1, \tau) + E \\ &A_2 - L_2[e^{-r\tau} - p(x, \tau)] + L_1[p(x, \tau) - p_1(x_1, \tau)] \geq E \end{aligned} \tag{20b}$$

Focusing on the second line in (20b), it should be clear that the premium of the new policyholders is (−1 times) the first bracketed expression on the left hand side of the inequality sign. The difference between A_2 and the premium must equal the new equity (E) since there is no other source of funds. Thus, the condition for writing the policies reduces to the following:

$$p(x, \tau; \sigma^2) - p(x_1, \tau; \sigma_1^2) \geq 0 \tag{21}$$

where σ_1^2, σ^2 = the risk parameters of the firm before and after the policy issue.

In general, if the firm is safer after the new policies are issued, i.e., if $p(x, \tau, \sigma^2) < p(x_1, \tau, \sigma_1^2)$, the stockholders will lose money on the transaction.[16] They will gain if the firm is more risky following the policy issue, so that $p(x, \tau, \sigma^2) > p(x_1, \tau, \sigma_1^2)$. Expression (20a) is satisfied as an equality only if the value of the put (per dollar of liabilities) is the same before and after the policy issue.

Unless the new policies are unusually risky or highly correlated with the old policies, the risk parameter of the firm after the policies are issued will be less than it was before due to the diversification effect. Since $\partial p(x, \tau)/\partial\sigma > 0$ and $\partial p(x, \tau)/\partial x < 0$, this implies that the firm can operate at a lower leverage ratio without expropriating value from the old policyholders.

This model may help to explain market behavior observed during insurance price and availability crises. For example, assume that the risk of policy class 2 is sufficiently high that $\sigma^2 > \sigma_1^2$. Then, in order to avoid expropriation, the leverage ratio must increase, leading to higher costs for the new policies. If there is an optimal leverage ratio (or range) and unexpected losses reduce the ratio to a suboptimal level, it may be difficult to restore the optimal ratio immediately. Expressions (20a), (20b), and (21) imply that the firm cannot raise the ratio without incurring a capital loss unless it charges more than the optimal premium to the new policyholders. Writing more business at a suboptimal leverage ratio may affect the reputation of the firm and therefore dampen future cash flows. Thus, the firm would prefer to write business at higher-than-market prices even if this means reducing its volume.[17]

[16] The put value is directly related to risk, i.e., $\partial p/\partial\sigma > 0$. The comparative statics of the Black-Scholes model are discussed in Ingersoll (1987).

[17] Some restriction on entry is necessary in order for firms to restore optimal leverage ratios by writing at higher-than-fair prices. New entry may be difficult in lines such as liability insurance due to information asymmetries, regulation, and other market imperfections.

The multi-class options model also has implications for the price of insurance. Consistent with the basic options model, the multi-class model also suggests that the price will be inversely related to the value of the insolvency put, i.e., safer firms should command higher prices. Empirical evidence that insurance prices are inversely related to the expected policyholder costs of insolvency is provided by Sommer (1996), Cummins and Danzon (1997), and Phillips, Cummins, and Allen (PCA) (1998).

19.5.4 Option Models and the Allocation of Equity Capital

A recent topic that has been addressed by several papers in the actuarial and financial literature is the allocation of equity capital (surplus) by line of business (e.g., Kneuer 1987, Butsic 1999, Merton and Perold 1993, Cummins, Phillips, and Allen 1998, Perold 1999, Myers and Read 1999, Cummins 2000). The usual objective in capital allocation is to assess a cost of capital charge to each line based upon the amount of capital assigned to the line and the riskiness of the line. The allocation of capital is motivated by the observation that holding capital in a financial institution is costly due to regulation, taxation, and agency costs. The general argument is that lines which consume more capital should bear a higher proportion of the firm's overall cost of capital than lines which consume less capital. Capital consumption is determined by the impact of the line of business on the insurer's insolvency put option.

In capital allocation, a typical objective is to attain a specified target level of the expected policyholder deficit (EPD) or insolvency put option. E.g., a firm may want to have an insolvency put value of no more than 5 percent of liabilities. The allocation of capital among lines in the multiple line firm is problematical because writing multiple lines leads to diversification effects whereby the amount of capital needed to attain the EPD target in the multiple line firm will be less than the sum of the capital needed to attain the target if each line were operated as a separate or "stand-alone" firm. The impact of diversification is non-linear in the option modeling context, and it is not obvious how to allocate the diversification effect.

Merton and Perold (1999) and Perold (1999) propose a marginal approach to allocating capital. To facilitate the discussion of their methodology, we consider a firm with three lines of business—labeled lines 1, 2, and 3. We assume that the multi-class option model presented in the preceding section is used in conducting the capital allocation. In this context, the M-P method of capital allocation is conducted in two steps: (1) Calculate the equity capital required to obtain the EPD target by firms that combine two of the businesses. There are three possible combinations: businesses 1&2, businesses 1&3, and businesses 2&3. (2) Calculate the marginal capital required to attain the target when the excluded business is added to the two-business firms, i.e., the marginal capital required if a firm consisting of two businesses were to add the third business. The capital allocated to a given business is equal to the marginal capital required when it is added to the appropriate two-business firm. Because the calculation is made for each two firm combination, the method provides a unique capital allocation for each of the business lines comprising the firm.[18]

Merton and Perold (1993) argue that capital allocations based on stand-alone capital are likely to lead to incorrect decisions about the projects undertaken by the firm and the performance evaluation of lines of business. They also argue that allocating all capital among lines may lead to the rejection of positive net present value projects. Their view is that the unallocated capital should be held at the "corporate" level rather than being charged to any specific division.

An alternative to the Merton and Perold approach which does allocate 100 percent of capital has been proposed by Myers and Read (M-R) (1999). They also use an option pricing model to allocate capital but reach different conclusions from Merton and Perold. Whereas Merton and Perold allocate capital at the margin by adding entire lines or division to the firm (a *macro* marginal allocation), Myers and Read allocate capital by determining the effect of very small changes in loss liabilities for each line of business (a *micro* marginal allocation). Myers and Read allocate capital by differentiating the insolvency put with respect to the amount of liabilities resulting from each line of business, essentially deriving the effect on the put of infinitesimal changes in the liabilities from each line. They argue that their approach leads to a unique allocation of the firm's entire capital across its lines of business.

Examples presented in Cummins (2000) indicate that the amounts of capital allocated to each line of business can differ *substantially* between the Merton-Perold and Myers-Read methods. Thus, the two methods will not yield the same pricing and project decisions. The Myers-Read method has considerable appeal because it avoids the problem of how to deal with the unallocated capital under the Merton-Perold approach. In addition, most decision making regarding pricing and underwriting *is* marginal in the sense of Myers and Read, i.e., typically involving very small changes to an existing portfolio. However, more research is needed to determine which model is more consistent with value maximization.

A different perspective on the multiple line firm problem is provided by Phillips, Cummins, and Allen (PCA) (1998). Unlike Merton-Perold and Myers-Read, they assume that no friction costs are present in the market for insurance. They derive the following formula for the market value of line i's claim on the insurer:

$$P_i = L_i e^{-(r_f - r_{L_i})\tau} - w_{L_i} B(A, L, \tau) \tag{22}$$

where P_i = the market value of line i's claim on the firm,

L_i = the nominal losses owed to line i,

r_f, r_{Li} = the risk-free rate and the liability inflation rate of line i,

[18] The order in which the businesses are combined into firms does not matter because all three two-business combinations are used, i.e., the allocated capital of each business is obtained on the assumption that two of the businesses have already been combined.

$w_{li} = L_i/L$,
A, L = total assets and total liabilities of the insurer, and
$B(A, L, \tau)$ = the insurer's overall insolvency put.

Or, in other words, the market value of line i's claim on the firm at time τ before the policy expiration date is equal to the nominal expected value of its loss liabilities at the expiration date $L_i e^{r_{L_i}\tau}$, discounted at the risk-free rate, minus the line's share of the firm's overall insolvency put option. Thus, the discount for insolvency risk in line i's claim on the firm depends upon the overall insolvency risk of the firm and not just on the line-specific levels of risk. Intuitively, this is because each line of business has access to the firm's entire capital in the event that losses are larger than expected.

One of the implications of the surplus allocation result in the PCA analysis is that the market value of the line-specific claims on the insurer should be equal after controlling for differences in line-specific growth rates and the overall risk level of the firm, regardless of differences in the risk characteristics of the individual lines of business. PCA test their theoretical prediction by comparing a measure of price for short-tail lines to the same measure for long-tail lines and show that prices are consistent with the predictions, i.e., after controlling for the overall insolvency risk of the firm and line-specific growth rates, there is no significant difference between the price measures for the short and long-tail lines.

A linkage can be developed between the PCA and the Myers-Read models of capital allocation in the presence of friction costs. That is, the price to be charged to line i would be equal to the market value of line i's claim on the firm from equation (22) plus the costs of the marginal capital that must be added to the firm to maintain a target insolvency put, where marginal capital is obtained using the formulas in Myers and Reed (1999). Further theoretical and empirical exploration of this approach could provide a new class of insurance option pricing models.

19.5.5 The Insurer as a Down-and-Out Option

One of the implications of the simple option model of the firm is that the equity owners can gain at the expense of the debt holders by increasing the risk of the firm (the derivative of a call option with respect to the risk parameter is positive). Nevertheless, in actual securities and insurance markets, stockholders usually do not exploit this feature of the call option. One way to explain this is through reputational arguments, as suggested above. Another approach is to examine penalties and restrictions that might be imposed on firms adopting expropriative strategies.

One type of restriction that is often used in bond markets is the safety covenant. For example, the bond agreement may specify that the firm will be reorganized if its value ever drops to a specified level. Although insurance contracts usually do not include safety covenants, regulation has a similar effect. Specifically, under the U.S. risk-based capital system, regulators are required to seize an insurer if its equity capital falls below a specified level $K > 0$ (see Cummins, Harrington, and Niehaus 1994). This regulatory "option" terminates the equity holders' claim in the firm if the difference between assets and liabilities ever reaches the boundary. Because the chance of reaching the boundary is an increasing function of risk, the risk-based capital system changes equity owner incentives with regard to risk-taking.

Risk-based capital can be modeled using a type of option known as the *down-and-out option* (see Merton 1973, Cox and Rubenstein 1985). Let $W(A, L, \tau)$ equal the value of a down-and-out call option on an insurer with assets A and face value of liabilities L. The time-to-expiration of the option is τ. Prior to τ if the value of the assets ever reaches the *knock-out boundary* $K = b\,L\,\exp(-\eta\tau)$, the stockholders' interest in the firm is terminated and the assets revert to the debt holders, where b and η are constants.

To analyze the insurance case, assume that $\eta = 0$. Then the knock-out boundary is constant, and the value of the firm reverts to the debt holders if assets fall to $b\,L$. Also assume that the call option has an infinite life, i.e., $\tau = \infty$.[19] The formula for the infinite down-and-out call is:

$$W(A, L) = A - bL\left(\frac{A}{bL}\right)^{-2r_f/\sigma^2} \tag{23}$$

where σ^2 = the dispersion parameter of the insurer. Because the value of an infinite-lived conventional call option equals the value of the assets (the value of an infinite-lived conventional put is zero), equation (23) implies that the value of the firm's debt is $D(A, L) = bL(A/bL)^{-\gamma}$, where $\gamma = 2r_f/\sigma^2$.

The effects of changes in risk on the equity and debt of the down-and-out firm are as follows:

$$\frac{\partial W}{\partial \sigma^2} = -\frac{\partial D}{\partial \sigma^2} = -bL\ln\left(\frac{bL}{A}\right)\left(-\frac{2r_f}{\sigma^4}\right) \tag{24}$$

Expression (24) is < 0 if $bL < A$. Thus, increases in risk reduce the equity holders' share of the value of the firm and increase the debt holders' share. Equity holders have an incentive not to increase risk because this increases the chance that their share in the firm will be forfeited to the debt holders due to the knock-out feature. This provides a useful model of the value of the firm under solvency regulation.

Like the standard Black-Scholes model, the down-and-out option can be generalized to multiple asset and liability classes. For one asset class and two types of liabilities, the value of the down-and-out option is given by equation (23), with dispersion parameter σ_n^2 (defined following equation (19)).

19 These assumptions are used to simplify the discussion. A closed form solution exists for finite-lived down-and-out options with $\eta > 0$. See Cox and Rubenstein (1985, p. 410).

If the liquidation rule allocates assets in proportion to nominal liabilities, the analysis of the firm's decision to accept new business is similar to the multi-variate Black-Scholes analysis discussed above. The firm will issue new business provided that the equity value of the firm after the policy (and stock) issue is greater than the value of the firm before the issue plus the amount of new capital raised. The condition is expressed as follows:

$$W(A_1 + A_2, L_1 + L_2) - W(A_1, L_1) \geq E$$
$$A_2 - bL_2\left(\frac{x}{b}\right)^{-2r_f/\sigma^2} - bL_1\left[x^{-2r_f/\sigma^2} - x_1^{-2r_f/\sigma_1^2}\right] \geq E \tag{25}$$

where $x = (A_1 + A_2)/(L_1 + L_2)$,
$x_1 = A_1/L_1$,
σ^2 = dispersion parameter after the policy issue, and
σ_1^2 = dispersion parameter before the policy issue.

Since new debt holders will pay the fair value of their coverage, there is neither a gain nor a loss in equity if the term in brackets in (25) is equal to zero. The dispersion of the firm typically will be lower following the policy issue. Thus, the firm should be able to operate at a lower leverage ratio after issuing new policies. The down-and-out model provides an alternative options interpretation of the insurance firm which may provide a better description of observed insurer behavior than the Black-Scholes model.

19.6 PRICING CAT CALL SPREADS AND BONDS

A number of new financial instruments have been introduced recently to accomplish the securitization of insurance risk. The securitization process involves the development of financial instruments whose payoffs are triggered by losses from hurricanes, earthquakes, oil spills and other contingent events traditionally financed through insurance. The most prominent insurance derivatives are catastrophic risk (CAT) call spreads and bonds, where the payoffs are triggered by losses from property catastrophes. CAT call spreads are option contracts that pay off on the basis of an industry loss index, while the payoff on most CAT bonds is triggered by the losses of the specific insurer issuing the bonds.

CAT futures were first introduced by the Chicago Board of Trade (CBOT) in 1992, call spreads were introduced in 1993, and a major design change was implemented in 1995. We abstract from most of the institutional details of the CBOT contracts and focus instead on the key mathematical features that enter into the pricing of this type of contract. Define an industry loss index, I, which is based on the value of losses from catastrophic events over a clearly defined period of time. For the CBOT contracts, the loss index is compiled by Property Claims Services (PCS), a statistical agent sponsored by the insurance industry, based on surveys of insurers following property catastrophes. The index is equal to the insured catastrophic property loss divided by \$100 million, e.g., a \$2 billion event would have an index value of 20. The CBOT securities are call spreads on the index, so that the payoff is defined as: $P = \text{Max}[0, I\text{-}M] - \text{Max}[0, I\text{-}U]$, where M = the lower strike price and U = the upper strike price of the option. E.g., a 20/40 spread would be triggered by an industry-wide loss of > \$2 billion and pay a maximum of 20 points for losses ≥\$4 billion, with each point worth \$200. The pricing of CBOT-type contracts has been investigated by Cummins and Geman (1995). They model the stochastic process representing CAT losses as having a continuous component, modeled as a geometric Brownian motion process, and a discrete component based on a Poisson jump model. They do not find a closed form expression for the option price but are able to price the contracts using Monte Carlo simulations.

To develop a more general model of CAT options, it would be necessary to address the problem of incomplete markets. Naik and Lee (1990) show that when jump risk is systematic (i.e., correlated with the market portfolio), the market is incomplete. Consequently, risk-neutral valuation techniques cannot be applied without imposing additional restrictions such as constant jump magnitudes (as in Cummins and Geman 1995) and non-systematic jump risk (as in Merton 1978). If such restrictions are unrealistic, it is necessary to resort to equilibrium pricing, resulting in utility dependent option values. Insurance pricing with jumps in the incomplete markets setting is an important area for future research.[20]

To price CAT bonds, we adopt Merton's (1978) approach to the incomplete markets problem, i.e., the assumption that catastrophic risk is non-systematic.[21] We again abstract from the institutional details and focus primarily on the mathematical structure. CAT bonds are debt instruments issued by an insurer and sold to investors. The investors contribute capital in exchange for the bonds. The capital is placed in a single purpose reinsurer (SPR) that exists solely to handle the CAT bond issue. The SPR is set up in the form of a trust that holds the proceeds of the bond issue. The bonds are invested in safe securities such as Treasury bonds. Because of the formation of the trust and the investment of proceeds in low risk securities, the SPR is virtually free of credit risk. The insurer agrees to pay interest on the bonds. However, repayment of principal is contingent on the insured event.[22] If the contingent event occurs, the bond covenant permits the insurer to withdraw funds from the trust

[20] Chang (1995) has developed an option pricing model incorporating jumps in both complete and incomplete markets settings.

[21] Evidence that property catastrophe losses are not correlated with returns on the stock market is provided in Litzenberger, Beaglehole, and Reynolds (1996) and Canter, Cole, and Sandor (1997).

[22] For example, the bond may call for full repayment of principal unless a hurricane occurs in a specified geographical region such as Florida that satisfies certain severity criteria such as the amount of insured property loss and/or physical severity criteria such as the Saffir-Simpson rating of the storm.

to pay losses arising from the event; and the bond holders forfeit some or all of their principal.

Because of the assumption that CAT risk is uncorrelated with the stock market, CAT bonds can be considered *zero-beta securities*. The zero-beta feature also suggests a simple pricing model for these securities. Specifically, in asset pricing theory, zero-beta securities should earn the risk-free rate. However, because there is some probability that the principal will not be repaid, the bond coupon rate on these securities should be sufficient to deliver the risk-free rate to investors after taking into account the potential loss of principal. Assuming a one-period bond, this suggests the following pricing model:

$$P(1+r_f) = P(1+r_c-\lambda) \tag{26}$$

where P = the bond principal,

r_c = the coupon rate on the bond, and

λ = the expected loss of principal due to an insured event expressed as a proportion of principal.

It is easy to see that the coupon rate should be: $r_c = r_f + \lambda$. Thus, to price the bond, one needs to estimate the expected loss from the contingent event.

Cummins, Lewis, and Phillips (CLP) (1999) provide an estimate of λ based upon the frequency and severity of hurricanes and earthquakes using both historical loss data provided by PCS and simulated losses from Risk Management Solutions (RMS), a modeling firm specializing in the simulation of catastrophic events. They estimate the expected loss for a contract covering the layer from \$25 to \$50 billion dollars in total industry losses to range from less than 1 percent to 2.4 percent, depending on the probability distributions selected to model the frequency and severity distributions and the data source (PCS or RMS). Thus, a CAT bond covering losses in this layer might be issued at the Treasury rate plus a maximum of 240 basis points. Actual CAT bond issues have generally been sold at higher margins above the Treasury rate, although the risk premia have declined over time as investors have become more familiar with these bonds.

19.7 CONTINUOUS TIME DISCOUNTED CASH FLOW MODELS

19.7.1 Certainty Model

Continuous time models for insurance pricing have been developed by Kraus and Ross (KR) (1982) and Cummins (1988). As an introduction, consider the Kraus-Ross continuous time model under conditions of certainty.

To simplify the discussion, assume that the current value of losses is determined by a draw from a random process at time 0. Loss payments occur at instantaneous rate θ, while loss inflation is at exponential rate ρ, and discounting is at rate r_f. The differential equation for the rate of change in outstanding losses at time t, in the absence of inflation, is: $dC_t/dt = -\theta C_t$. Solving this equation for C_t yields the amount of unpaid claims at any given time (the reserve): $C_t = C_0\, e^{-\theta t}$. Thus, the assumption is that the claims runoff follows an exponential decay process with average time to payout $= 1/\theta$.

Considering inflation (π), the rate of claim outflow at any given time is: $L_t = \theta C_t e^{\pi t}$. The premium is the present value of losses, obtained as follows:

$$P = \int_0^\infty L_t e^{-r_f t} dt = \int_0^\infty \theta C_0 e^{(\pi - \theta - r_f)t} dt = \frac{\theta C_0}{r_f + \theta - \pi} \tag{27}$$

In (27), π could be >, =, or < economy-wide inflation. The model also can be used to estimate the market value of reserves, R_τ:

$$R_\tau = \int_\tau^\infty \theta L_0 e^{-(\theta + r_f - \pi)t} dt = \frac{\theta L_0}{\theta + r_f - \pi} e^{-(\theta + r_f - \pi)\tau} \tag{28}$$

19.7.2 Uncertainty Models

Kraus and Ross also introduce a continuous time model under uncertainty. This model is based on arbitrage pricing theory (APT). The KR model allows for market-related uncertainty in both frequency and severity.

The following differential equation governs the claims process: $dC/dt = \alpha_t - \theta C_t$, where α_t = accident frequency. The frequency process affects the evolution of outstanding claims for a period of length T (the policy period). After that point, no new claims can be filed. During the entire period (0 to ∞) claims inflation takes place according to the price index q_t. The parameters α_t and q_t are governed by the k economic factors of arbitrage pricing theory. These factors are modeled as diffusion processes:

$$dx_i = m_i x_i dt + \sigma_i x_i dz_i, \quad i = 1, 2, \ldots, k. \tag{29}$$

The parameters are log-linear functions of the factors, e.g.:

$$\log(q) = \sum_{i=1}^{k} q_i \log(x_i) + \log(q_0) \tag{30}$$

where q_0 = the price level of the average claim at policy inception.

Arbitrage pricing theory implies that the value of outstanding claims at any time t, $V(x, C, t)$, where x is the vector consisting of the x_i, is governed by the following differential equation:

$$E\left[\frac{dV}{V}\right]+\left[\frac{\theta qC}{V}-r_f\right]dt=\sum_{i=1}^{k}\lambda_i\sigma_i\left[Cov\left(\frac{dV}{V},\frac{dx_i}{x_i}\right)\Big/Var\left(\frac{dx_i}{x_i}\right)\right]dt \tag{31}$$

where λ_i = the market price of risk for factor $i = (r_{mi} - r_f)/\sigma_i$, and
r_{mi} = the market return on a portfolio that is perfectly correlated with the ith risk factor.

The premium formula is obtained by applying the multivariate version of Ito's lemma (see Ingersoll 1987) and then solving the resulting differential equation. The formula is:

$$P=\left(\frac{\theta\alpha_0 q_0 L_0}{\rho+\theta}\right)\left[\frac{1-e^{-\rho_\alpha\tau}}{\rho}\right] \tag{32}$$

where $\rho = r_f - \pi - \Sigma_i\ \lambda_i\sigma_i q_i$,
$\rho_\alpha = r_f - \pi_\alpha - \Sigma_i\ \lambda_i\sigma_i(q_i + \alpha_i)$,
$\pi = \Sigma_i\ [.5\sigma^2 q_i(q_i - 1) + q_i m_i]$,
$\pi_\alpha = \Sigma_i\ [.5\sigma^2(\alpha_i + q_i(\alpha_i + q_i - 1) + (\alpha_i + q_i)m_i]$.

The premium given by (32) is similar to the premium for the certainty case except for the presence of the market risk loadings (λ_i terms) in the denominator. These loadings are the company's reward for bearing systematic risk. The α_i and q_i are the "beta coefficients" of the model.

For the company to receive a positive reward for risk bearing, the risk loading term must be negative, i.e., losses must be negatively correlated with some of the market factors such that the net loading is < 0. The model requires estimates of the market prices of risk for the k risk factors as well as the beta coefficients for insurance. This would be difficult given the available data. Like most other financial pricing models for insurance, this model gives the price for an insurance policy that is free of default risk.

A continuous time model that prices default risk has been developed by Cummins (1988). Assets and liabilities follow geometric Brownian motion:

$$\begin{aligned} dA &= (\alpha_A A - \theta L)dt + A\sigma_A dz_A \\ dL &= (\alpha_L L - \theta L)dt + L\sigma_L dz_L \end{aligned} \tag{33}$$

where α_A, α_L = asset and liability drift parameters,
σ_A, σ_L = asset and liability risk (diffusion) parameters,
A, L = stock of assets and liabilities,
θ = the claims runoff parameter, and
$dz_A(t)$, $dz_L(t)$ = possibly correlated standard Brownian motion processes.

The asset and liability processes are related as follows: $\rho_{AL} = Cov(dz_A, dz_L)$.

The model is more realistic than the standard options model since it does not have a fixed expiration date but rather allows the liabilities to run off over an infinite time horizon, i.e., it models liabilities as a perpetuity subject to exponential decay.[23] Cummins uses the model to obtain the market value of default risk, $D(A, L)$. Using Ito's lemma to differentiate D and then using either a hedging argument or the ICAPM to eliminate the risk terms, one obtains the confluent hypergeometric differential equation. The solution is:

$$D(x) = \frac{\Gamma(2)}{\Gamma(2+a)} b^a x^{-a} e^{-b/x} M(2, 2+a, b/x) \tag{34}$$

where $a = 2(r_r + \theta)/Q$, $b = 2\theta/Q$, $Q = \sigma_A^2 + \sigma_L^2 - 2\,\sigma_A\,\sigma_L\,\rho_{AL}$, and M = Kummer's function (see Abramowitz and Stegun 1972).

This perpetuity model has significant potential for pricing blocks of policies subject to default risk. It poses easier estimation problems than the Kraus-Ross model since one need only estimate the variance and covariance parameters of assets and liabilities rather than betas and factor risk premia.

19.7.3 Pricing Multiple Claim Insurance Contracts

Shimko (1992) develops an equilibrium valuation model for insurance policies which extends the prior literature in three important ways. First, his model explicitly recognizes the non-linear payoff structures resulting from the deductibles and maximum policy limits found in many insurance policies. Second, both the frequency and severity of losses are allowed to vary systematically. By contrast, many of the option pricing models of insurance assume the liabilities of the insurer evolve as smooth geometric Brownian motion, which essentially combines these two features into one process. Third, Shimko's model allows for multiple claims over the lifetime of the policy.

Shimko assumes the claim amount C, conditional upon a claim being filed, for an individual will follow a geometric Brownian motion process:

$$dC_t = \alpha_c C_t dt + \sigma_c C_t dZ_{ct}. \tag{35}$$

To incorporate the deductible and policy limit provisions, the payoff to the policy-holder conditional upon a claim being filed at time t will equal

$$S_t = \min[\max(C_t - D, 0), M] \tag{36}$$

[23] A perpetual put option model incorporating jumps that also might be applicable to insurance pricing has been developed by Gerber and Shiu (1998).

where S_t is the payoff, D is the deductible, and M is the policy limit. The arrival of claims is modeled as a non-stationary Poisson process where the expected intensity of claims arrival is a geometric diffusion process equal to

$$d\lambda_t = \alpha_\lambda \lambda_t dt + \sigma_\lambda \lambda_t dZ_{\lambda t} \tag{37}$$

where the constant term α_λ measures the non-stochastic expected growth in claim frequency over the time period while σ_λ is the instantaneous volatility. The model allows for correlation between the claim arrival and amount processes where $dZ_{ct}dZ_{\lambda t} = \rho_{c\lambda}dt$ is the instantaneous correlation between C and λ.

To solve for the value of the insurance policy, V, Shimko considers two cases. In the first case he simplifies the problem and assumes the policy has a maximum indemnity payment M equal to positive infinity and a positive deductible D. Invoking the ICAPM equilibrium pricing relationships discussed earlier, he finds a closed form solution for the policy value V. The formula is quite complicated and therefore is not presented here. However, there is an intuitive interpretation which is useful to discuss.

The value of the policy is given by

$$V(C,\lambda,\tau;D) = \lambda W(C,\tau;D) \tag{38}$$

where W represents the expected payout by the insurance company conditional on a claim being filed by the policyholder and λ is the expected number of claims. Shimko shows that W is equal to the fair value of the cash flows needed to replicate the cash flows of the insurance contract. The replicating cash flows are as follows: (1) At the beginning of the policy period, if the claim amount is greater than the deductible, the insurer must purchase a risky perpetuity that pays Cdt and must sell a risk-free perpetuity that pays Ddt. If $C < D$ at time zero, do nothing. (2) Over the policy period the insurer must continuously revise the position. Whenever $C > D$, the insurer must go long in the risky perpetuity and short the risk-free perpetuity; otherwise hold nothing. (3) At the end of the policy period, the insurer must liquidate its positions.

To solve for the more general case when the policy includes a per-claim maximum indemnity limit, M, the revised valuation formula is[24]

$$Y(C,\lambda,\tau;D,M) = \lambda W(C,\tau;D) - \lambda W(C,\tau;M+D). \tag{39}$$

The intuition behind this result is readily apparent after we rewrite equation (36) as

$$S_t = \min[\max(C_t - D,0), M] = \max(C_t - D,0) - \max[C_t - (D+M),0] \tag{40}$$

[24] Readers familiar with Shimko's paper will note this formula differs from his equation for the value of the insurance policy with a maximum indemnity limit shown on page 235. The difference arises as we define the maximum indemnity payment M to be the largest payment the insurance company will make to its policyholder. This interpretation of a policy limit is standard in the insurance literature.

Thus, the payout to the policyholder is truncated from above as the policyholder takes a short position in a second insurance policy with a deducible equal to $(D + M)$, relieving the insurer from paying large losses.

The model which Shimko develops has a number of interesting implications. First, consider the case when there is no deductible, i.e., $D = 0$. In this is case, the cash flows needed to replicate the payoffs on the insurance contract are quite simple. Whenever $C > 0$ at time zero, the insurer must purchase a risky perpetuity that continuously pays λCdt over the term of the contract and will be liquidated at policy termination. The fair value of this cash flow is

$$V(C,\lambda,\tau; D=0) = \frac{\lambda C}{\delta} - e^{-\delta\tau}\frac{\lambda C}{\delta} \tag{41}$$

where τ is the term of the insurance policy and δ is the risky discount rate. If there is no correlation between the claims arrival and/or the conditional claim amount processes and the market portfolio, the discount rate δ will only be a function of the risk-free rate of interest and the expected growth rates of the claims arrival and amount processes, α_λ and α_c. Thus, when there is no deductible and no market risk, there is no reward for underwriting risky liability payments. In addition, increases or decreases in the riskiness of the claims arrival and/or amount processes will have no effect on the fair value of the insurance policy. Only positive correlation between the loss processes and the market portfolio will be priced in the contract. Thus, In the absence of market risk, a risk premium based upon the volatility of the liability processes cannot be justified.

When a positive deductible is introduced into the model, greater levels of volatility will increase the value of the insurance contract. When both a positive deductible and a policy limit are introduced, the effect of increasing volatility on the value of the insurance contract is ambiguous. On one hand, increasing levels of volatility will increase the value of the policyholder's long position in the first risky perpetuity in (41). However, the increased volatility also makes it more likely the policy limit will be reached, increasing the value of the short perpetuity in (41). Either effect can dominate.

19.8 CONCLUSIONS

This paper discusses the principal financial pricing models that have been developed for property-liability insurance and proposes some extensions. Insurance pricing models have been developed based on the capital asset pricing model, the intertemporal capital asset pricing model, arbitrage pricing theory, and options pricing theory. The models assume either that insurance policies are priced in accordance with principles of market equilibrium or minimally that arbitrage opportunities are avoided.

Additional research is needed to develop more realistic insurance pricing models. For example, most of the models assume that interest rates are non-stochastic even though insurers face significant interest rate risk. Modeling multiple-line firms with multi-period claim runoffs also poses challenging problems. With few exceptions, existing financial models do not price the risk of insolvency. Estimation problems, especially for betas and market risk premia, are a major problem given the existing insurance data. Option models and perpetuity models may offer solutions to some of these problems, since they rely on relatively few parameters and can be modified to incorporate stochastic interest rates. However, the options models often rely upon market completeness and no-arbitrage arguments which are difficult to justify for some insurance contracts. Additional research is needed on the pricing of insurance in incomplete markets.

In addition to models now in existence, models based on multi-factor asset pricing theory (Fama and French 1993, 1996), martingale pricing (Duffie 1988), and lattice modeling (Boyle 1988) may provide promising avenues for future research. Modifications of the perfect information, perfect markets results for information asymmetries also will become increasingly important as the field continues to advance. Also, future work which incorporates frictions in capital markets (Froot and Stein 1998) may add additional insights into the behavior of prices in insurance markets. We also expect to see further advances in pricing models and concepts for CAT bonds and options as the market for these innovative products continues to develop. Finally, researchers have applied fuzzy set theory (FST) to financial pricing (e.g., Cummins and Derrig 1997, Young 1996). We did not explore this topic in the present paper because providing an explanation of fuzzy mathematics sufficient for readers to understand the application would require too much space. However, FST may be a promising approach to explore in future research because it provides a rigorous set of rules for incorporating vague or imprecise information (e.g., expert judgment, etc.) into insurance ratemaking. Thus, FST has the potential to add another dimension to the standard financial pricing techniques, which implicitly assume a degree of precision in the information used in pricing that is rarely realized in practice.

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