Statistical and Financial Models of Insurance Pricing and the Insurance Firm

J. David Cummins

Although the field of risk and insurance is sometimes said to lack a central paradigm, several paradigms exist that define the foundations of the discipline. Two of the most important are: (1) the pure statistical model of insurance risk pools, which originates in probability theory and actuarial science; and (3) the financial model of the insurance firm and insurance pricing that draws upon modern financial theory. The two paradigms taken together constitute an important component of the core of the field of risk and insurance or insurance economics. Insurance demand theory and applications of various branches of economics such as industrial organization also are important elements of this core.

To a significant extent the existing strands of insurance research have emerged in parallel, and few serious attempts at integration have been carried out. One reason for this is that each area is highly specialized and technical. Insurance-oriented textbooks that successfully bridge the gap between the specialist and a wider audience do not exist. In order for the field of risk and insurance to progress more rapidly, this obstacle must be overcome. The objective of this paper is to take an initial step in that direction by providing an introduction to the central results of two of the principal strands of insurance research: statistical and financial models of insurance pricing and the insurance firm.

The next section considers statistical models. In spite of the critical importance of risk pooling in insurance, the statistical foundations of this phenomenon are not well-understood by many insurance experts. Misstatements and fallacies regarding pooling, laws of large numbers, the central limit theorem, and related concepts are pervasive, ranging from principles of insurance courses in the classroom to insurance tax cases in the federal

---

* J. David Cummins is Harry J. Loman Professor of Insurance and Risk Management and Executive Director of the Huebner Foundation at the Wharton School, University of Pennsylvania.

The author is grateful to Stephen D'Arcy, Richard A. Derrig, and an anonymous referee for helpful comments on an earlier draft of this article. A portion of this material was presented as Dr. Cummins' Pedogogical Seminar at the American Risk and Insurance Association meeting at Reno, Nevada in August 1989.

courtroom. This problem exists even though the basic statistical concepts are relatively simple and can be understood and appreciated without a high level of mathematical training. Section I attempts to correct this problem by discussing the implications of the law of large numbers and central limit theorem for insurance risk pooling and by analyzing some of the more persistent insurer's risk fallacies. The discussion then turns to collective risk theory, which provides the most sophisticated statistical model of insurance. This field, which originated primarily in Europe, has experienced some recent advances that significantly increase its potential for practical application.

A later section of the paper focuses on financial models of insurance. Financial modelling is one of the most rapidly growing fields of insurance research, particularly on the international scene. The paper provides an introduction to the basic financial applications in insurance. This discussion covers the insurance capital asset pricing model (CAPM), discounted cash flow modeling, and options modeling. Duration, immunization, and asset/liability management also are discussed. The emphasis throughout is on theoretical developments, but the reader should be aware that a growing empirical literature also exists, providing a possible topic for a future review article.

Major progress in understanding insurance pricing and insurance markets can be achieved through the integration of the statistical and financial theories of insurance pricing and insurance firms. If this article provides the foundation of knowledge to assist researchers in beginning the integration process, it has accomplished its objective.

Statistical Models of Insurance

Individual Risk Theory

The Model. A basic model that provides some important insights into the pooling process is individual risk theory (see Cummins (1974), Bowers, et al. (1986)). The random variable analyzed is the total monetary amount of claims arising from an insurance pool during a specified reporting period (e.g., one year). The total amount of claims is the sum of the claims of the individual exposure units comprising the pool:

$$S_N = \sum_{i=1}^{N} X_i \tag{1}$$

where S_N = total amount of claims in a specified period of time,
X_i = claims of exposure unit i, and
N = number of exposure units comprising the pool.

Well-known theorems regarding sums of random variables can be used to obtain the moments of S_N. To conserve notation, the exposure units are assumed to be identically distributed. [1] Specifically,

---

[1] The assumption that risks are identically distributed has been adopted frequently in the literature as a mathematical and notational convenience. Unfortunately, some readers have

$$E(S_N) = \mu_N = N\mu$$

$$Var(S_N) = \sigma^2_N = N\sigma^2 + 2\sum_{j=2}^{N}\sum_{i=1}^{j-1}\sigma_{ij} \quad (2)$$

where μ = mean loss per exposure unit,
μ_N = mean loss of the pool,
σ^2 = variance of loss of each exposure unit,
σ_N^2 = variance of loss of the pool, and
σ_{ij} = covariance of the ith and jth exposure units.

If the exposure units are uncorrelated, the covariance term in (2) vanishes.

Analogous formulas apply to weighted sums of random variables. E.g., suppose that $S_N = \Sigma_i a_i X_i$, where a_i are constants, i = 1, . . . , N. Also assume that the X_i are not identically distributed. Then the mean and variance of S_N are:

$$E(S_N) = \mu_N = E[\sum_{i=1}^{N} a_i X_i] = \sum_{i=1}^{N} a_i \mu_i \quad (3)$$

$$Var(S_N) = \sigma^2_N = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \sigma_{ij}$$

where $\mu_i = E(X_i)$ and $\sigma_i^2 = Var(X_i)$. Although constraints may be placed on the a_i in specific contexts, mathematically they do not have to be positive, sum to 1, or have absolute values less than 1. Among other applications, the weighted mean and variance formulas are useful in solving problems involving quota share reinsurance.

The Law of Large Numbers and the Central Limit Theorem. The law of large numbers and the central limit theorem play important roles in individual risk theory. The law of large numbers, in particular, provides a critical statistical foundation for the existence of insurance. Unfortunately, the law of large numbers and central limit theorem are often misunderstood. One reason for the confusion is that there are numerous versions of these important theorems. Thus, referring to *the* law of large numbers or *the* central limit theorem can be misleading. Nevertheless, all versions of the theorems carry the same qualitative implications for insurance.

In discussing the law of large numbers it is helpful to refer to Chebyshev's inequality:

> *Chebyshev's inequality:* Let random variable X have a distribution function with finite variance σ^2 and finite mean μ. Then for every $k > 0$,

$$Pr[|X-\mu| < k\sigma] \geq 1 - 1/k^2$$

The inequality states that a realization of random variable X will be within k standard deviations of its mean with probability at least as large as $1 - 1/k^2$. The inequality applies for any probability distribution satisfying the conditions of the theorem (finite means and variances). For many probability distributions, the probability that X will fall within k standard deviations of

---

extrapolated this assumption into the inference that identically distributed risks are *necessary* for the existence of insurance. As shown below, this is not the case.

its mean is actually higher than $1 - 1/k^2$. For example, consider the probability that a realization of the random variable X will fall within 2 standard deviations of the mean (i.e., k = 2). The Chebyshev inequality tells us that the probability of being within 2 standard deviations is at least .75. If X is normally distributed, the probability is .9545. Thus, while the inequality is valuable in theoretical work, it is not sufficiently precise for most empirical applications.

Using the Chebyshev inequality, it is possible to prove the following theorem:

> *Law of Large Numbers: Version 1.* Let random variable Y be equal to the number of successes in N repetitions of an experiment with probability of success equal to p. (I.e., the probability distribution of Y is the binomial.) Define the relative frequency of success as Y/N. Then for every $\epsilon > 0$,

$$\lim_{N\to\infty} Pr[|\tfrac{Y}{N}-p| < \epsilon] = 1 \tag{4}$$

According to the theorem, the probability that the observed proportion of successes will be arbitrarily close to the probability of success (p) approaches 1 as the number of trials approaches infinity. Thus, in a very large number of coin tosses, the probability that the observed proportion of heads will deviate from 1/2 approaches zero as the number of trials (N) becomes large.

To prove the law of large numbers, use Chebyshev's inequality. First, rewrite (4) as follows:

$$Pr[|Y-Np| < N\epsilon] = Pr[|Y-\mu| < N\epsilon] \tag{5a}$$

where μ = Np is the mean of the binomial distribution of Y. From Chebyshev, we know that

$$Pr[|Y-\mu| < k\sigma] \geq 1 - 1/k^2 \tag{5b}$$

It is apparent that the Chebyshev result applies to (5a) if $k\sigma = N\epsilon$ or $k = N\epsilon/\sigma$. Combining (5a) and (5b), we can write:

$$Pr[|Y-\mu|/N < \epsilon] = Pr[|Y-\mu| < N\epsilon] \geq 1 - \sigma^2/(N^2\epsilon^2) \tag{5c}$$

Allowing N to approach infinity yields the desired result.

A more general version of the law of large numbers that does not require the binomial assumption is the following:

> *Law of Large Numbers: Version 2.* Let $X_1, \ldots, X_N$ be a random sample from a distribution with finite mean μ and finite variance σ^2. Then

$$\lim_{N\to\infty} Pr[|\overline{X}-\mu| < \epsilon] = 1 \tag{6}$$

where $\overline{X} = \Sigma_i X_i/N$ = the sample mean.

This version indicates that the sample mean will be arbitrarily close to the true distributional mean with probability approaching 1 as N approaches infinity. The proof is straightforward using Chebyshev's inequality. This version of the law of large numbers is often used in defining insurer's risk (see below).

It is important to recognize what the law of large numbers does and does not imply. First, the law does not require that random variables be normally distributed, only that means and variances be finite so that the Chebyshev theorem can be used.[2] Secondly, the law does not imply the following:

$$\lim_{N \to \infty} Pr[|\sum_{i=1}^{N} X_i - N\mu| < \epsilon] = 1 \tag{7}$$

for ϵ arbitrarily small. I.e., even though the sample mean is arbitrarily close to the distributional mean with a high probability as N goes to infinity, the sum of the random observations (X_i) does not become (with high probability) arbitrarily close to $N\mu$. This point is critically important in understanding insurer's risk and is discussed in more detail below.

The Central Limit Theorem. Let $X_i, \ldots, X_N$ be a random sample from a probability distribution with mean μ and variance σ^2. Define the random variable Y, with probability distribution $F_N(y)$:

$$Y = \frac{\sum_{i=1}^{N} X_i - N\mu}{\sigma\sqrt{N}} = \sqrt{N}\left(\frac{\overline{X} - \mu}{\sigma}\right) \tag{8}$$

Then, as N approaches infinity, the probability distribution $F_N(y)$ approaches N(y), where N(y) is the standard normal distribution.[3]

The central limit theorem provides important insights into the concept of insurer's risk. However, it is of limited practical relevance in working with insurance loss distributions because the distributions usually encountered in practice approach normality so slowly that they remain significantly skewed even in large insurance pools.

Insurer's Risk. There has been much confusion in the insurance economics literature with regard to the implications of the law of large numbers and the central limit theorem. This section states the correct implications and discusses some of the most persistent fallacies.

Insurer's risk can be defined in many ways. However, two important definitions are characterized as insurer's relative risk and insurer's absolute risk. To define insurer's relative risk, consider the insurer's risk model in equation (1) and define the following:

$$\textit{Mean Loss: } \overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \tag{9a}$$

$$\textit{Variance of } \overline{X}\textit{: } Var(\overline{X}) = \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N} \tag{9b}$$

---

[2] Actually, the result is even more general and holds for distributions with non-existent (infinite) variances, as long as the mean is finite. See Hogg and Craig (1978).

[3] Some of the formalities needed to make the theorem completely rigorous have been omitted. See Hogg and Craig (1978).

$$\textit{Standard Error of } \overline{X}: \sigma_{\overline{x}} = \frac{\sigma}{\sqrt{N}} \tag{9c}$$

The variance result in (9b) uses the formula for weighted sums of random variables, recognizing that the weights in the formula for the mean are $a_i = 1/N$, all i. The X_i are assumed to be independent and identically distributed.

Using the law of large numbers (version 2), we can write:

$$\begin{aligned} \lim_{N\to\infty} Pr[|\overline{X}-\mu| < \epsilon] &= \lim_{N\to\infty} Pr[|\overline{X}-\mu| < k\frac{\sigma}{\sqrt{N}}] \\ &\geq \lim_{N\to\infty} [1 - \frac{\sigma^2}{\epsilon^2 N}] = 1 \end{aligned} \tag{10}$$

Thus, the average loss per exposure unit becomes arbitrarily close to the true mean of the loss distribution with probability approaching 1 as N approaches infinity. This suggests two definitions of insurer's relative risk:

Insurer's Relative Risk Version 1: $IRR_1 = \sigma/\sqrt{N}$

Insurer's Relative Risk Version 2: $IRR_2 = \sigma/(\mu\sqrt{N})$

IRR_1 is just the standard error of the mean, while IRR_2 is the ratio of the standard error of the mean to the distributional mean loss per exposure unit.[4]

Both definitions of relative risk have the important property that they go to zero as N goes to infinity. Thus, relative risk becomes negligible in large pools.

The same relationships can be used to define absolute risk. Adapting (10) by multiplying through within the probability statements by N, one obtains:

$$\begin{aligned} &\lim_{N\to\infty} Pr[|\sum_{i=1}^{N} X_i - N\mu| < N\epsilon] \\ &= \lim_{N\to\infty} Pr[|\sum_{i=1}^{N} X_i - N\mu| < k\sigma\sqrt{N}] = 1 \end{aligned} \tag{11}$$

This suggests the following definition of insurer's absolute risk:

Insurer's Absolute Risk: $IAR = \sigma\sqrt{N}$

It is clear that absolute risk becomes infinite as the size of the pool (N) approaches infinity. Thus, the probable deviation of the total loss of the insurer from its expected value becomes infinite (for any given epsilon) as N goes to infinity, and insurer's absolute risk cannot be said to disappear in large pools.

These results suggest that one must be precise in defining risk when discussing risk reduction through pooling. Risk is reduced in the relative but not the absolute sense as the pool grows. Thus, common statements in the economics and finance literature that "non-systematic risk disappears in large markets" or "the market is sufficiently large that risk can be disregarded" are meaningful in one sense but not in others. Some but not all types of risk disappear as the number of exposure units increases.

---

[4] IRR_2 is also called the coefficient of variation.

The concepts of absolute and relative risk have implications for the amount of surplus required to maintain an acceptable probability of ruin for an insurance pool. The required surplus is often called the *buffer fund.* To illustrate the concept of the buffer fund, assume that the limiting result implied by the central limit theorem applies exactly so we can write:

$$Pr[|\frac{S_N - N\mu}{\sigma\sqrt{N}}| < y_\alpha] = Pr[|S_N - N\mu| < y_\alpha \sigma\sqrt{N}] = 1-\alpha \tag{12}$$

where y_α = the standard normal deviate such that $N(y_\alpha) = 1-\alpha$. Thus, to be assured with probability $1-\alpha$ that all claims will be paid, the company must have on hand a total buffer fund of $y_\alpha\ \sigma\sqrt{N}$. The buffer fund per policy is: $y_\alpha\ \sigma\sqrt{N}/N = y_\alpha\ \sigma/\sqrt{N}$.

The total buffer fund is a function of absolute risk, while the buffer fund per policy is a function of relative risk. Thus, the insurer's total buffer fund goes to infinity with pool size (N), while the required buffer fund per policy goes to zero. Again, one must be precise in defining the resources required to operate the insurance pool.

An example of risk reduction through pooling is presented in Table 1. The table shows the effects of pooling independent, identically distributed risks. The mean, variance, and standard deviation per risk are given in the first panel of the table. The severity mean and variance are of the same magnitude as the auto liability means and variances presented in Cummins and Wiltbank (1983). The frequency mean was chosen for illustrative purposes and implies that an insured can expect to have an accident about once in 7 years. The mean and variance of the total claims distribution are calculated from the frequency and severity means and variances using formulas given in Cummins and Wiltbank (1983) and repeated in the note to Table 1. It is assumed that the total claims distribution is normal.[5]

The table shows the effects of pooling as the pool grows in size from 100 to 100 million exposure units. The pool mean increases proportionately with the size of the pool, as does the pool variance. However, the pool standard deviation (insurer's absolute risk) increases less than proportionately with N. Nevertheless, the IAR and consequently the required buffer fund (for a ruin probability of 1 percent) grow with pool size. Because they grow less than proportionately, the buffer fund per policy declines as the pool grows, illustrating the distinction between absolute and relative risk.

The buffer fund may be obtained directly from policyholders, from the equity markets, or in some combination. In a pure one-period mutual, the fund would be collected entirely from policyholders. In an inter-generational mutual, the fund would be partly obtained from present policyholders and partly from prior policyholders; while in a stock company, equity contributions plus retained earnings would make up the buffer fund.

---

[5] This assumption is made for illustrative purposes only and is not meant to imply that the pure premium distribution actually is normal.

Table 1

Risk Reduction Through Pooling

	Severity	Frequency	Total Claims	
Mean =	993	0.15	148.95	
Variance =	2,500,000	0.18	552,488.82	
Std Dev =	1,581.14	0.42	743.30	
N	**Pool Mean**	**Pool Std Dev**	**Buffer Fund**	**BF/Policy**
100	14,895	7,433	17,319	173.188
1,000	148,950	23,505	54,767	54.767
10,000	1,489,500	74,330	173,188	17.319
100,000	14,895,000	235,051	574,668	5.477
1,000,000	148,950,000	743,296	1,731,879	1.732
10,000,000	1,489,500,000	2,350,508	5,476,684	0.548
100,000,000	14,895,000,000	7,432,959	17,318,795	0.173

Note: Var(Total Claims) = Var(Severity)*Mean(Frequency) + Var(Frequency)*Mean(Severity)^2
Pool Mean = Mean(Total Claims)*N
Pool Std Dev = Sqrt(N*Var(Total Claims))
Buffer Fund = 2.33*(Pool Std Dev)
BF/Policy = Buffer Fund/N

Whatever the source of equity, policyholders must pay for the cost of the capital (equity) supporting the pool because they benefit from the security (reduction in ruin probability) that the capital provides. Since the amount of capital per policy does disappear in large pools, it is correct to conclude that the risk charge (cost of capital) is negligible and policies can be priced approximately at the expected value of loss. However, the total resources devoted to the insurance enterprise would still be substantial.

It is a fallacy to assert that the total buffer fund goes to zero in a large pool, using the model defined above. However, it is interesting to inquire whether it is ever true that a pool would need no equity in the limit. The answer, suggested by Samuelson (1963), is yes, under specific circumstances.

In particular, consider a pool with a fixed expected loss, E. The pool insures homogeneous exposure units with means μ and variances σ^2.[6] The pool retains proportion α of each risk and reinsures the balance. The expected loss of the pool is $E = N\alpha\mu$. Since E is fixed, the number of units written will be $E/\alpha\mu$ and the variance will be:

$$\sigma^2_N = \sum_{i=1}^{\frac{E}{\alpha\mu}} \alpha^2 \sigma^2 = \frac{E}{\alpha\mu}\alpha^2\sigma^2 = \frac{E}{\mu}\alpha\sigma^2 \tag{13}$$

[6]Homogeneity is assumed merely for convenience and is not necessary in order for the no-surplus result to hold.

Consequently, by allowing N to go to infinity (as α goes to 0), the risk of the pool vanishes and no buffer fund is required.

Samuelson calls this the "$1/\sqrt{N}$ Law." The (absolute) risk of the pool is not reduced by adding risks but by *subdividing* them. The stock market provides an example of a mechanism for reducing risk through subdivision, since shareholders own small fractions of the companies in their portfolios.

A numerical example of the $1/\sqrt{N}$ Law is provided in Table 2. The table shows the effects of increasing the size of the pool and simultaneously reducing the share of the loss liability retained by the pool (alpha) in such a way that the expected loss remains constant. The mean is held constant at the level that would be attained if the pool consisted of 100 risks with alpha of 1. The table indicates dramatically the effects of subdividing or quota-sharing risks. The total buffer fund and the buffer fund per policy both approach zero as the pool grows, showing the effects of diversification on "non-systematic" risk.

Table 2

Subdividing Risks:The 1/Sqrt(N) Law

	Severity	Frequency	Total Claims		
Mean =	993	0.15	148.95		
Variance =	2,500,000	0.18	552,488.82		
Std Dev =	1,581.14	0.42	743.30		
Alpha	**N**	**Pool Mean**	**Pool Std Dev**	**Buffer Fund**	**BF/Policy**
1	100	14,895	7,433	17,319	173.1879
0.1	1,000	14,895	2,351	5,477	5.4767
0.01	10,000	14,895	743	1,732	0.1732
0.001	100,000	14,895	235	548	0.0055
0.0001	1,000,000	14,895	74	173	0.0002
0.00001	10,000,000	14,895	24	55	5.477E-06
1E-06	100,000,000	14,895	7	17	1.732N-07

Note: Var(Total Claims) = Var(Severity)*Mean(Frequency) + Var(Frequency)*Mean(Severity)²
Pool Mean = Mean(Total Claims)*N*Alpha
Pool Std Dev = Sqrt(N*Var(Total Claims)*Alpha²)
Buffer Fund = 2.33*(Pool Std Dev)
BF/Policy = Buffer Fund/N

The $1/\sqrt{N}$ Law does not lead to the elimination of risk if the losses of the exposure units comprising the portfolio are not independent. To see this, assume that $\sigma_{ij} = \rho\sigma_i\sigma_j = \rho\sigma^2$, all i and j, i.e., exposure units are identically distributed (so $\sigma_i = \sigma_j$) and all pairs of exposure units have the same correlation coefficient (ρ).[7] Using the formula for the variance of weighted

[7] If all units have the same correlation coefficient, ρ must be > 0.

sums of random variables (equation (3)) with $a_i = \alpha$, all i, the variance of the pool is:

$$Var(S_N) = N\alpha^2\sigma^2 + N(N-1)\alpha^2\rho\sigma^2 \tag{14}$$

Again holding the mean constant at $E = N\alpha\mu$, and allowing α to go to zero (i.e., $N \rightarrow \infty$):

$$\sigma_N^2 = \alpha^2 \frac{E}{\alpha\mu} \sigma^2 + \alpha^2\rho\sigma^2\left[\frac{E^2}{\alpha^2\mu^2} - \frac{E}{\alpha\mu}\right] \tag{15}$$

$$= \alpha \frac{E}{\mu} \sigma^2(1-\rho) + \rho\sigma^2 \frac{E^2}{\mu^2}$$

In this case, the variance does not vanish even with infinite subdivision, but rather approaches $\rho\sigma^2 E^2/\mu^2$, a function of the common covariance between units. This is a type of "capital asset pricing" relationship, where only non-diversifiable or covariance risk is relevant.

The Homogeneity Fallacy. A persistent fallacy is that risks must be homogeneous in order for pooling to be operable. This fallacy is prevalent in insurance textbooks. The following statement appears in one prominent text:

> The first requirement [for insurability of risk] is the existence of a large number of homogeneous exposure units. The purpose of this requirement is to enable the insurer to predict based on the law of large numbers. . . . a large number of heterogeneous exposure units would not fulfill this requirement . . . If heterogeneous exposure units were grouped together, it would be difficult for the insurer to predict accurately the average frequency and severity.

This is another example of the need to be precise when discussing pooling. Homogeneity usually is desirable but is not necessarily required for insurability. A more compelling reason for grouping risks is to prevent adverse selection (see Rothschild and Stiglitz (1976), Cummins, et al. (1984)). However, the latter rationale has little to do with pooling or the predictability of total losses.

We first demonstrate that homogeneity is not necessarily desirable by giving an example of a case in which it is undesirable (Feller (1968)). The example applies to Bernoulli distributed exposure units, i.e., to the case where losses of the pool are binomially distributed.

> *Theorem.* Assume that a pool consists of N independent exposure units, each with Bernoulli frequency distributions with parameters p_i, $i = 1, 2, \ldots, N$. The amount of loss is the same for each exposure unit and non-stochastic, and the average parameter is $\bar{p} = \Sigma_i\, p_i/N$. For a given value of the average parameter, the variance of the pool is maximized if the exposure units are homogeneous.

Proof: Without loss of generality, assume the loss amount is 1. Then the ith exposure unit is characterized by the following probability distribution:

Loss	Probability
0	$1-p_i$
1	p_i

If x_i is the loss amount of unit i, then $Var(x_i) = p_i(1-p_i)$, and the variance of the pool is:

$$\sigma^2 = \sum_{i=1}^{N} p_i(1-p_i) = \sum_{i=1}^{N} (\bar{p}+p_i-\bar{p})(1-\bar{p}-p_i+\bar{p}) \qquad (16)$$

Letting $d_i = p_i - \bar{p}$ and multiplying the two parenthetical expressions in (16), we have:

$$\begin{aligned}\sigma^2 &= \sum_{i=1}^{N} [\bar{p}(1-\bar{p})+(1-\bar{p})d_1-\bar{p}d_1-d_i^2] \\ &= \sum_{i=1}^{N} [\bar{p}(1-\bar{p})-d^2_i] = N\bar{p}(1-\bar{p})-\sum_{i=1}^{N} d^2_i\end{aligned} \qquad (17)$$

The terms involving a constant (p or $1-\bar{p}$) times d_i drop out because $\Sigma_i d_i = 0$. Since $d_i^2 \geq 0$, it is apparent from the last line in (16) that σ^2 is maximized when $d_i = 0$, all i. QED

This result holds under the conditions stated but not under more general conditions. Nevertheless, it is revealing that homogeneity may under some circumstances be disadvantageous. The example has particular force in insurance because of the importance of the binomial model.

As an illustration of a situation where heterogeneity does not matter, consider an insurer that writes two types of risks, "good" risks and "bad" risks. The good risks have means and variances equal to .75 times the exposure unit pure premium mean and variance in Table 1, while the bad risks have means and variances equal to 1.25 times the pure premium mean and variance in Table 1. Exactly 50 percent of the risks written by the insurer are good risks and 50 percent are bad risks. Then the total buffer fund requirements for a 1 percent ruin probability are identical to those shown in Table 1. This is because the pool mean and variance are the same. E.g., the pool variance is: $.5*N*(.75\ \sigma^2) + .5*N*(1.25\ \sigma^2) = N\ \sigma^2$. Since the exposure unit total claims distributions are normal, the distribution of the sum is also normal and the buffer fund calculations can be done as before. This example illustrates that one must be very precise in talking about the effects of heterogeneity because some types of heterogeneity increase risk and other types do not.

Heterogeneity would increase risk in the above example if the proportions of good and bad risks were not known to be exactly 50 percent but rather the probability that any given risk accepted by the insurer is a good risk is 50 percent. This the type of heterogeneity introduces another source of risk and thus can significantly increase the risk of the pool and therefore the required buffer fund.[8] In most situations of this type pooling is still operable, but a larger pool is required to attain a given buffer fund charge per policy.

More general central limit theorems exist giving the conditions under which sums of non-identical random variables approach normality (see, for example, Cramer (1946, pp. 215-216)). If these conditions are met, one can use the theorem to make statements about absolute risk, relative risk, and buffer funds just as in the case where risks are identically distributed. There is a wide

---

[8] See the discussion of structure functions in Beard, et al. (1984) for a formal model of this type of heterogeneity.

class of cases in which homogeneity is not required in order for pooling to be effective.

Central limit theorems also are available for random variables that are neither identically distributed nor independent. Assume that a pool consists of N non-independent random variables to which a central limit theorem applies such that the sum S_N is normally distributed with mean μ_N and variance:

$$\sigma_N^2 = \sum_{i=1}^{N} \sigma_i^2 + 2\sum_{j=2}^{N} \sum_{i=1}^{j-1} \sigma_{ij} \tag{18}$$

The insurer will charge each risk a premium equal to the average mean, μ_N/N plus the required buffer fund divided by the number of policyholders.[9] The buffer fund charge per policy will be:

$$y_\alpha \frac{\sqrt{N\sigma_m + N(N-1)\sigma_{ijm}}}{N}$$

$$\text{where } \sigma_m = \text{the average variance} = \frac{\sum_{i=1}^{N} \sigma_i^2}{N}, \text{ and} \tag{19}$$

$$\sigma_{ijm} = \text{the average covariance} = \frac{2\sum_{j=2}^{N} \sum_{i=1}^{j-1} \sigma_{ij}}{N(N-1)}$$

Allowing N to go to infinity, the buffer fund charge becomes:

$$y_\alpha \sqrt{\sigma_{ijm}}$$

Thus, the buffer fund charge per policy does not go to zero, even in the limit.

Like the "$1/\sqrt{N}$" result for correlated risks, this covariance result is analogous to the capital asset pricing model in the sense that systematic (non-diversifiable) risk is priced even with "complete" diversification. Of course, the diversification is not really complete because we have considered only insurance risk and not asset-market risk. If the economy were characterized by complete markets, diversification across all assets might well eliminate the "undiversifiable" component of insurance risk.

Large Deviations. Brockett (1983) has pointed out a potential problem in using the normal distribution to analyze probability of loss distributions. This is the problem of "large deviations," discussed in Feller (1971, pp. 548-553). The problem is well-known to risk theorists, who have tried for years to find a convenient way to approximate the tails of insurance probability distributions.

Consider again the normalized sum, $(S_N - N\mu)/\sigma\sqrt{N}$. The central limit theorem asserts that this sum tends to normality as $N \rightarrow \infty$, i.e., $F_N(x) \rightarrow N(x)$, where $F_N(x)$ is the distribution function of S_N and N(x) is the standard normal distribution function. The limiting behavior is useful for moderate x, but for large x both $F_N(x)$ and N(x) are close to 1 and the central limit theorem ceases to convey useful information. What is really needed is information on the

---

[9]This premium assessment rule is not necessary to achieve the illustrated result but is introduced merely to simplify the discussion.

ratio $[1-F_N(x)]/[1-N(x)]$ in situations where both x and n tend to infinity. Various methods have been suggested for solving this problem, some of which are discussed in Brockett (1983) and Feller (1971).

It is important to understand the distinction between the large deviations problem and the buffer fund problem. The buffer fund problem uses the probability statement:

$$Pr\left[\frac{S_N - N\mu}{\sigma\sqrt{N}} < y_\alpha\right] \tag{20}$$

The large deviations problem is concerned with probability statements such as the following:

$$Pr\left[\frac{S_N - N\mu}{\sigma\sqrt{N}} < \frac{Nc - N\mu}{\sigma\sqrt{N}}\right] = Pr\left[\frac{S_N - N\mu}{\sigma\sqrt{N}} < \left(\frac{c-\mu}{\sigma}\right)\sqrt{N}\right] \tag{21}$$

As (21) makes clear, the large deviations problem involves a "moving target." This differs from the usual problem in risk pooling, where the target, e.g., y_α, is fixed.

Even with a fixed target, the normal distribution is not recommended for most practical applications, at least without testing other approaches. The reason is that the approach to normality may be very slow, so that the normal distribution underestimates the tail even for very large n. The normal distribution is useful primarily to help elucidate the nature of risk pooling. Methods that give more accurate solutions to insurance modelling problems are discussed below.

This section concludes with a brief discussion of the stable family of distributions. A distribution R is a member of the stable family if sums of random variables with distribution R have distributions that differ from R only by location and scale parameters (Feller (1971, pp. 169–173)). The stable family serves as a natural generalization of the normal distribution. The normal is the only member of the stable family that has a finite variance. Distributions without finite variances are frequently encountered in insurance (e.g., Paulson and Faris (1985), Cummins, et al. (1991)). An important aspect of the stable family is that if the sum of identically distributed independent random variables has a limiting distribution, that distribution is a member of the stable family. No other distributions occur as such limits. This makes the stable family extremely important for modelling insurance loss distributions.

Paulson (e.g., Paulson and Faris (1985)) has applied the stable family extensively in studying insurance claim distributions. He has found that risks such as oil spills (in gallons) tend towards stable distributions with infinite variances and in some cases infinite means (i.e., with characteristic exponent < 1).[10] As Feller (1971, p. 172) points out, this implies "that the average [of n

[10] The characteristic exponent is the parameter that differentiates members distributions of the stable family. The normal distribution has a characteristic exponent of 2. For characteristic

stable variables with characteristic exponent < 1] is likely to be considerably larger than any given component X_k." The important implication for insurance is that such risks are not poolable, i.e., there is no gain in predictability from insuring N rather than 1 such unit. Surprisingly, distributions with existent means but infinite variances may be poolable. Thus, pooling does break down in some instances but is much more robust than is usually recognized.

Collective Risk Theory

An alternative stochastic model of the insurance firm is collective risk theory (see Beard, Pentikainen and Pesonen (1984) or Buhlmann (1970)). This approach was developed initially by European actuaries and has attained a high degree of sophistication. This paper provides an overview of the collective risk model and summarize some of the major developments of the past few years.

The Collective Risk Model. Collective risk theory differs from individual risk theory in that it considers the insurance pool as a collective rather than looking at individual exposure units. It is not necessary to know the stochastic characteristics of the individual exposure units in the pool. The random process generating claim costs is broken down into two components: (1) the number of claims (frequency) and (2) the amount of loss per claim (severity). The total claims distribution is obtained by developing models of the probability distributions of frequency and severity and compounding these distributions to obtain the distribution of total claims.

The amount of claims during a particular measurement period is given by:

$$X = \sum_{i=1}^{N} y_i \tag{22}$$

where X = total claims,

y_i = the amount of the ith claim (i.e., severity), and
N = the number of claims (frequency).

Notice that the number of exposure units does not appear in this formula. Equation (22) is a random sum because the upper limit of summation, the number of claims (frequency), is a random variable. This contrasts with the individual risk theory model (equation (1)), where the upper limit of summation, the number of exposure units, is known.

The probability distribution of severity that generates the y_i represents the collective severity distribution of the pool. A collective distribution can even be used for large pools consisting of many different heterogeneous types of risks. In this case, the severity distribution would be a mixture of the individual severity distributions applicable to the underlying classes.[11] Of

---

exponents less than 2, the variance does not exist; and for characteristics exponents less than 1, the mean does not exist.

[11] Excellent discussions of severity mixture distributions are provided in Hogg and Klugman (1987) and McDonald and Butler (1987).

course, to use the model in a predictive sense, the mixture would have to remain relatively constant across periods. This criterion seems to be satisfied with sufficient accuracy in many real-world risk pools.

A multi-variate version of the collective risk model has been developed by Cummins and Wiltbank (1983, 1984). This model generalizes the standard model by permitting separate distributions for accident frequency and claims frequency. A given accident is permitted to give rise to some number n of claims, n = 0, 1, ... Each claim then contributes a severity observation. Different accident processes can be introduced, each with its own severity distribution. The multi-variate collective risk model is helpful in a number of ways, e.g., to determine the effects of different reinsurance retention limits in different parts of the collective.

The collective risk theory model is considerably more powerful than the individual risk theory model in analyzing real-world risk pools. The model is used to study reinsurance decision making, establish surplus levels, and measure ruin probabilities. The model also has an advantage over the individual risk model in terms of estimation. One year's data provides N observations on the frequency distribution, where N is the number of exposure units, and n observations on the severity distribution, where n is the number of claims. Thus, for a moderate-size pool, very good estimates of the frequency and severity distributions can usually be obtained. This is usually more accurate than trying to estimate the total claims distribution directly from exposure unit data. If loss severities and frequencies cannot be traced to individual exposure units, the latter method would be infeasible unless a long time-series of loss data were available and the total claims distribution had been reasonably stationary over time.

The frequency and severity distributions are compounded to obtain the total claims distribution. The model is given below:

$$F(x) = \sum_{i=0}^{\infty} p_k S^{k*}(x) \tag{23}$$

where F(x) = the distribution function of total claims,

$S_k{}^*(x)$ = the k-th convolution of the severity distribution function, i.e., the distribution of the sum of k severity random variables, and

p_k = the probability of k claims, where $p_k = p(k)$ is the frequency distribution.

In practice, the compounding process can be difficult. For only a few combinations of frequency and severity distributions does a closed form solution of (23) exist. Unfortunately, these tractable combinations of distributions rarely provide adequate models for real-world risk pools.

As an example of a closed form solution to (23), consider the case of geometric frequency and exponential severity:

$$\begin{aligned} &\textit{Frequency: } p(k) = p\, q^k,\ k = 0,1,2,... \\ &\textit{Severity: } \quad s(x) = \lambda e^{-\lambda x},\ x \leq 0 \end{aligned} \tag{24}$$

where p, q = parameters of the geometric frequency distribution, q = 1 − p,
λ = parameter of the exponential severity distribution.

Substituting equations (24) in (23) and simplifying,[12] one obtains:

$$F(x) = 1 - qe^{-\lambda px},\ 0 \leq x \leq \infty \tag{25}$$

Although the exponential is rarely a viable severity distribution, this example is interesting because it illustrates (23) and is useful for pedagogical purposes.[13]

Researchers have developed a number of different methods for studying F(x) in situations where the compounding process cannot be carried out explicitly. These fall into two categories: (1) approximation methods, and (2) "exact" methods. We consider the approximation methods first.

Approximation Methods for F(x). The approximation methods generally utilize the moments of F(x). The moments are easy to compute from the moments of the frequency and severity distributions even in cases where F(x) itself cannot be obtained in closed form. The first three moments (i.e., the mean, the variance, and the third moment about the mean), are adequate for most approximation methods, although researchers sometimes use the kurtosis of F(x), which requires the fourth moment about the mean. Formulas for the first three moments are provided in Cummins and Wiltbank (1983).

A convenient way to obtain higher moments of F(x) is to use the cumulant generating function:[14]

$$K_x(t) = K_K[K_Y(t)] \tag{26}$$

where $K_X(t)$ = cumulant generating function of the total claims distribution,
$K_K(t)$ = cumulant generating function of the frequency distribution,
$K_Y(t)$ = cumulant generating function of the severity distribution.

The cumulant generating function is the natural log of the moment generating function: $K_Z(t) = \ln[M_Z(t)]$, where $M_Z(t) = E(e^{tz})$ = the moment generating function of Z with auxiliary parameter t.

Moments are obtained by differentiating the cumulant generating function with respect to the auxiliary parameter t, setting t = 0, and then evaluating. The first three cumulants are the mean, variance, and third moment about the mean. Higher moments are also related to the higher cumulants, and moments can easily be obtained from these relationships if the cumulants are known. The reason for working with the cumulant generating function rather than the

---

[12] In simplifying, one uses the fact that convolutions of exponential distributions are gamma distributed, so that $S^{k*}(x)$ in (23) is gamma with scale parameter λ and shape parameter K. Another hint is to separate the term involving p_0 from those involving p_i, $i > 0$, before attempting to simplify. Also note that $S^{0*}(x) = 1$ for $x \geq 0$ and $= 0$ for $x < 0$.

[13] Notice that F(x) has a spike at zero, i.e., F(0) = 1 − q. Technically, such a spike is present whenever zero claims is a possible outcome; but in practice the probability of no claims in most pools is negligible and thus for practical purposes $F(0) \approx 0$.

[14] The form of the cumulant generating function given here is based on the moment generating function. The more general form is based on the characteristic function. The latter involves the use of complex numbers but exists in cases where the moment generating function does not exist.

moment generating function is that there is a substantial reduction in the algebra involved in the differentiation especially for higher moments.

Once the moments of F(x) have been obtained, the approximation methods can be used to estimate quantile points in the tail of the distribution. These are useful in estimating maximum probable losses (MPYs), ruin probabilities, and other quantities needed in risk management. Approximation methods involving more than two moments are needed because total claims distributions tend to be positively skewed, even in large pools.

One of the simplest approximation methods is the normal power method (see Beard, Pentikainen, and Pesonen (1984)). The three moment form is:

$$X_\alpha = \mu_x + \sigma_x[y_\alpha + \frac{\gamma_x}{6}(y_\alpha^2 - 1)] \tag{27}$$

where X_α = the quantile point of F(x) such that $\Pr(x \leq X_\alpha) = 1-\alpha$,
y_α = the α quantile point of the standard normal,
μ_X, σ_X = the mean and standard deviation of F(x), and
γ_X = skewness of F(x) = $E[(x-\mu_X)^3]/\sigma_X^3$.

The normal power formula adjusts the familiar normal approximation for the presence of skewness. Thus, the first two terms in (27) are identical to the normal approximation, while the third term is a function of the skewness (γ_X). This method is easy to use because it relies on the probability points of the standard normal distribution. It has been shown to be accurate for values of skewness ≤ 2, and conservative (i.e., giving values of X_α that are too high) for skewness > 2.[15]

Another approximation technique is the gamma approximation. This method approximates the tail of F(x) using the tail of a gamma distribution with the same skewness as F(x). A three parameter gamma is used in order to match the first three moments of F(x):

$$g(x) = \frac{1}{\Gamma(\alpha)\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} e^{-(\frac{x-\gamma}{\beta})}, \ x > \gamma \tag{28}$$

where g(x) = the three-parameter gamma density function. The first three moments are:

$$E(x) = \gamma + \alpha\beta \tag{29a}$$

$$E[(x-\mu)^2] = \sigma^2 = \alpha\beta^2 \tag{29b}$$

$$\gamma_x = E\left[\frac{(x-\mu)^3}{\sigma^3}\right] = \frac{2}{\sqrt{\alpha}} \tag{29c}$$

The approximation is expressed as: $F(x) \approx G(x)$. The approximation begins by obtaining the standardized random variable $z = (x-\mu)/\sigma$, i.e., $x = z\sigma + \mu$. To facilitate calculations, it is desirable to work with the standardized gamma random variate $y = (x-\gamma)/\beta$, so $G(x) = H[(x-\gamma)/\beta]$, where:

[15] For further discussion, see Beard, Pentikainen, and Pesonen (1984).

$$H\left(\frac{x-\gamma}{\beta}\right) = \int_0^{\frac{x-\gamma}{\beta}} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy \qquad (30)$$

Noticing that $(x-\gamma)/\beta = z\sqrt{\alpha}+\alpha$, and substituting the latter expression as the upper limit of integration of (30) gives a convenient form of the approximation:

$$G(x) = H(z\sqrt{\alpha}+\alpha) = \int_0^{z\sqrt{\alpha}+\alpha} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy \qquad (31)$$

Thus, to approximate the quantile point of F(x) corresponding to a specific value x, estimate the parameters, compute the standardized value z, and apply equation (31).

"Exact" Methods. The normal power, gamma, and other approximation formulae have the virtue of convenience but are inaccurate in some practical applications. Consequently, researchers have sought more accurate ways of approximating F(x). Two very important methods have been developed that solve this problem in different ways: (1) Fourier inversion of the characteristic function of F(x) and (2) the Panjer recursion formula. The Fourier method has been extensively tested by Paulson (e.g., Paulson and Dixit (1989)), while the recursion formula was developed by Panjer (e.g., Panjer (1981)). A complete discussion of these methods would be beyond the scope of the present paper. However, the methods are outlined briefly below to provide some basic intuition.

The Fourier transform method is based on the one-to-one correspondence that exists between the probability distribution function of a random variable and its characteristic function. The characteristic function is defined as follows:

$$\Phi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x) \qquad (32)$$

where i = the imaginary unit, satisfying the equation $i^2 = -1$.

The characteristic function is the complex generalization of the moment generating function. The advantage of the characteristic function is that it always exists, whereas the moment generating function does not exist for some probability distributions. The characteristic function is also called the inverse Fourier transform of f(x).[16]

A random sum such as equation (22) has a particularly convenient characteristic function:[17]

$$\Phi(t) = \sum_{i=0}^{\infty} p(k)\, \Phi_S(t)^k = M_k[ln(\Phi_S(t))] \qquad (33)$$

[16] The Fourier transform usually is defined with -t rather than t as the auxiliary parameter, but there is not actual difference since t is an arbitrary constant.

[17] An analogous result holds for the moment generating function, when this function exists.

where $M_k[.]$ = the moment generating function of frequency, and
$\Phi_S(t)$ = the characteristic function of severity.

Thus, the characteristic function of F(x) is the moment generating function of frequency with auxiliary parameter $\ln(\Phi_S(t))$ in place of the usual parameter t.[18] For the Poisson frequency distribution with parameter λ, we have:

$$\Phi(t) = e^{-\lambda + \lambda\Phi_S(t)} \tag{34}$$

The use of characteristic functions provides a method for calculating F(x) in cases where severity convolutions and/or the direct compounding of frequency and severity are not feasible. The solution makes use of *Fourier inversion* of the characteristic function. The Fourier inversion formula is:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx}\, \Phi(t)\, dt \tag{35}$$

By first obtaining and then inverting the characteristic function, one can obtain f(x). The use of a computational algorithm known as the Fast Fourier Transform (FFT) makes this process more manageable by greatly reducing computational time.

Fourier analysis is extremely powerful and could eventually revolutionize the study of probability of loss distributions. Acceptance of the method has been slowed by the relatively advanced nature of the mathematics it entails and by the absence of computer programs that have been adapted for convenient use in insurance applications. As these barriers are overcome, use of the method to analyze problems in insurance will become more common.

An equally ingenious method for calculating the total claims distribution has been developed by Panjer (1981). This method can be used when the frequency distribution satisfies the following recursion relationship:

$$p(k) = (a + \frac{b}{k})p(k-1) \tag{36}$$

where p(k) is the probability of k claims and a and b are constants. This is not a very restrictive requirement since the two distributions used most commonly in property-liability insurance, the Poisson and the negative binomial, both satisfy the relationship.

Panjer's recursion formula requires a discrete severity distribution. However, continuous severity distributions can be easily and accurately "discretized" for use in the formula (see Gerber (1982), Panjer and Lutek (1983)).

The distribution function of total claims F(x) is defined above (equation (23)). Panjer's insight was that, using (36), the density function of total claims f(x) can be rewritten as:

---

[18] It is assumed that the moment generating function of frequency exists. Ln is this formula stands for the complex logarithmic function, which is the inverse of the exponential function.

$$f(x) = \sum_{j=1}^{Min(x,m)} \left(a + \frac{jb}{x}\right) s(j)\, f(x-j) \tag{37}$$

where m = the maximum attainable value of the discrete severity distribution, and

s(j) = the discrete or discretized severity distribution.

The method works by starting with f(0), which equals p(0), inserting it on the right hand side of (37) and then computing f(1). The values of f(0) and f(1) then are used to compute f(2), and so on. Proceeding in this way, the entire f(x) distribution eventually can be calculated. The method is more efficient computationally than Fourier inversion for smaller pools but less efficient for larger pools. However, both methods can be used effectively for many applications.

Modelling the Reserve Runoff

Neither individual nor collective risk theory traditionally considers the claims runoff tail. This is an obvious omission, which must be remedied in order to provide realistic models of insurance loss processes.

In the United States, actuaries traditionally have modelled the tail using reserve development ratios (also called the chain ladder method (see Lemaire (1985)). These ratios express the relationship between incurred losses at various times after the start of the accident year. Thus, there is usually a ratio to develop losses from 15 months to 27 months, from 27 months to 39 months, etc. The ratios are computed as averages of the relevant relationships for prior accident years.

The chain ladder method is essentially flawed. It does not take into account economic factors that may affect reserve development. Instead, it merely assumes that the past will be repeated in the future. Perhaps more fundamentally, it is often based on incurred rather than paid losses and thus incorporates past reserve errors. Since these "errors" often reflect deliberate managerial judgments, the model does not capture the economic or statistical nature of the true runoff process. Reserve estimation should be based on paid not incurred claims.

Actuaries have begun to recognize the defects of the incurred chain-ladder method and now use a variety of methods testing both the paid and incurred runoff. A good introduction to American reserving methods is provided in Casualty Actuarial Society (1990).

A significant advance in the analysis of paid claim runoffs was the development of the Taylor separation method (Taylor (1977)). This method separates the payout proportions (proportions of total claims paid each development period) from the inflation factors characterizing each calendar year. It facilitates forecasting by permitting the application of projected inflation factors while recognizing the relative stability of the payout proportions.

More sophisticated versions of the separation method have been developed since the publication of Taylor's 1977 article (see Taylor (1986) and Lemaire

(1985)). However, the essential features of the method can be illustrated by reference to the original model. The model works with a paid claim development matrix such as the following:

Accident	**Development Years:**				
Year	**1**	**2**	**3**	**4**	**5**
XXXX	c_{01}	c_{02}	c_{03}	c_{04}	c_5
XXXX + 1	c_{11}	c_{12}	c_{13}	c_{14}	
XXXX + 2	c_{21}	c_{22}	c_{23}		
XXXX + 3	c_{31}	c_{32}			
XXXX + 4	c_{41}				

In the matrix, c_{ij} is the amount paid in development year j for claims arising in accident year xxxx+i. Taylor's original formulation expresses the c_{ij} as payments per claim so that the entries in the matrix are total payments divided by the number of claims for each accident year.[19]

To simplify the discussion, the above matrix incorporates the assumption that all claims are paid in five development years and that no data are available other than the runoff triangle given in the matrix. In practice, many lines of insurance have longer runoff periods and information would be available on prior years that could be used to estimate the runoff and inflation factors.

Taylor's insight was to recognize that each element in the payout matrix reflects the reserve runoff process as well as the price level prevailing in each calendar year. He decomposed each element in the matrix into these two factors. The decomposition matrix is shown below:

Accident	**Development Years:**				
Year	**1**	**2**	**3**	**4**	**5**
XXXX	$r_1 k_0$	$r_2 k_1$	$r_3 k_2$	$r_4 k_3$	$r_5 k_4$
XXXX + 1	$r_1 k_1$	$r_2 k_2$	$r_3 k_3$	$r_4 k_4$	
XXXX + 2	$r_1 k_2$	$r_2 k_3$	$r_3 k_4$		
XXXX + 3	$r_1 k_3$	$r_2 k_4$			
XXXX + 4	$r_1 k_4$				

In this matrix, r_i is the proportion of total claims paid during development period i, and k_j is the "price factor" for year xxxx+j. The successive diagonals of the matrix represent the same calendar year and thus are characterized by the same price factor.

Taylor suggested an ingenious technique for estimating the parameters of the decomposition matrix. To illustrate, assume that data are available on the

---

[19] The payments in each row are divided by the same estimate of the ultimate number of paid claims. Thus, summing across a row after all claims have been paid yields an estimate of average claim severity for the accident year represented by the row.

c_{ij}, where the data elements are denoted $\hat{c}_{ij}$. The estimate of k_4 is obtained by summing along the main diagonal of the runoff matrix as follows:

$$\sum_{i=1}^{5} r_i k_4 = k_4 = \hat{c}_{41} + \hat{c}_{32} + \hat{c}_{23} + \hat{c}_{14} + \hat{c}_{05}$$

The result relies on the fact that the r_i sum to 1. The estimate of k_4 is used to estimate r_5, as follows: $\hat{r}_5 = \hat{c}_{05}/\hat{k}_4$, where $\hat{k}_4$ and $\hat{r}_5$ are the estimated values of k_4 and r_5. The parameter estimate $\hat{k}_3$ is obtained by summing across the diagonal containing the k_3 terms and using the relationship $r_1 + r_2 + r_3 + r_4 = 1 - r_5$, etc.

After estimating all of the r_i and k_j, reserves are computed by estimating the k_j factors for future calendar years and using these in conjunction with the estimated r_i to fill out the blank portion of the runoff triangle. For example, the reserve established at the end of year xxxx + 4 for claims still to be paid for accident year xxxx + 1 is $\hat{r}_5 k_5^f$ where k_5^f is the estimated price index factor for year xxxx + 5.

An example of the Taylor separation method is provided in Table 3. The table is based on the industry-wide automobile liability payout matrix from *Best's Aggregates and Averages: 1989*. The payouts have been deflated approximately for growth in the number of claims.[20] In the example, all claims are assumed to be paid during a six year runoff period, i.e., the reserves outstanding after five years are assumed to be paid in the sixth runoff year.

To obtain the estimated price factor for 1988 ($\hat{k}_5$), sum across the main diagonal of the matrix. The result, shown in the table, is 30743.36. This number is divided into the entry for accident year 1983, payment year 6, to obtain an estimate of $r_5 = 1687/30743 = 0.055$. To obtain an estimate of the price factor for 1987 ($\hat{k}_4$), sum across the diagonal beginning with 1987, payment year 1, and ending with 1983, payment year 5, and then divide this sum by $(1 - \hat{r}_5)$ to obtain 27811.29. Then $\hat{r}_4$ is the sum of the payments in payment year 5 for accident years 1983 and 1984 divided by the sum of $\hat{r}_4$ and $\hat{r}_5$, i.e., $(1162 + 1510)/(30743 + 27811) = 0.046$, etc.

The Taylor method and its generalizations are far superior to incurred loss development techniques for reserve estimation. In addition, they are more compatible with the discounted cash flow analysis inherent in the financial models discussed below.

Financial Models of the Insurance Firm

Insurance exists as a method for dealing with particular types of financial risk and for providing services associated with the management of these risks. Insurance institutions have been created as an efficient way to perform these risk sharing and risk management functions. The structure of insurance

[20] Specifically, an approximate index of the number of claims was obtained from various issues of *Best's Aggregates and Averages.* The payout amounts for accident years 1984 through 1988 were divided by the index values. The result normalizes the payouts in terms of the number of claims in 1983. This adjustment is approximate and is for illustrative purposes only. In an actual application, more precise estimates of the number of claims would be required.

Table 3

Taylor Separation Method
Decumulated Payout Matrix: Deflated for Claim # Growth

Accident Year	Payment Year: 1	2	3	4	5	6
1983	8679	7486	3815	2240	1162	1687
1984	9083	8122	4251	2444	1510	
1985	9245	8528	4394	2889		
1986	9211	8785	4831			
1987	9500	9269				
1988	10558					

Payout Lag(i)	r(i)	Year	Price Factors	Price Index
1	0.348	1983	24957.42	1.000
2	0.308	1984	25257.50	1.012
3	0.155	1985	26120.82	1.047
4	0.089	1986	26936.32	1.079
5	0.046	1987	27811.29	1.114
6	0.055	1988	30743.36	1.232

institutions and the insurance industry is determined by market forces and is linked to financial markets, markets for goods and services, and the overall structure of the economy. Insurance prices and the financial and organizational structure of insurance companies are determined by a complex set of supply and demand relationships. It is within this overall context that the analysis of financial models of the insurance firm takes place.

In the financial approach, both financial structure and price are endogenous to insurance and financial markets. Insurers are participants in these markets and survive in the long-run by finding the most efficient set of prices, contracts, investment portfolios, and other structural elements. The financial view can provide an explanation of organizational form, capital structure, dividend policy, and other aspects of insurance markets and institutions.

The application of financial models to insurance problems is a rapidly growing field. Excellent introductory discussions are provided in Smith (1986) and D'Arcy and Doherty (1988). More technical discussions that review and extend financial models of insurance are provided in Cummins (1990b) and (1991). To minimize the degree of overlap with those sources, this discussion provides an introduction to the central results; readers are referred to Cummins and Harrington (1987) and Cummins (1990b) and (1991) for more detailed discussions and more advanced models.

A Simple Model of the Insurance Firm

A simple financial model of the insurance firm can be used to provide a number of important insights into the management of insurance risk pools. The rudiments of this model were first presented in Ferrari (1968). The model is based on the following equation:

$$Y = I + \pi_U = r_A A + r_U P \tag{38}$$

where Y = net income,
I = investment income (net of expenses),
π_U = underwriting profit (premiums less expenses and losses),
A = assets,
P = premiums,
r_A = rate of investment return on assets, and
r_U = rate of underwriting return (proportion of premiums).

Equation (38) can be expressed as a return on equity as follows:

$$r_E = \frac{Y}{E} = r_A \frac{A}{E} + r_U \frac{P}{E} = r_A \left(\frac{L}{E} + 1\right) + r_U \frac{P}{E} = r_A(ks+1) + sr_U \tag{39}$$

where E = equity (policyholders' surplus),
L = liabilities = A − E,
s = P/E = premiums-to-surplus ratio, and
k = L/P = liabilities-to-premiums ratio (funds generating factor).

Equation (39) indicates that the rate of return on equity for an insurer is composed of a component due to investment income and a component due to underwriting income. The investment income part consists of the rate of return on assets multiplied by a leverage factor, (ks + 1), which is a function of the premiums-to-surplus ratio and the funds generating factor. The latter approximates the average time between policy issue and claims payment or the average holding period. The underwriting return component is the product of the underwriting profit ratio and the premiums-to-surplus ratio. Thus, the insurer is a leveraged firm where the leverage is achieved by issuing insurance policies.

Equation (39) can be written in an interesting way as follows:

$$r_E = r_A + s(r_A k + r_U) \tag{40}$$

This equation shows that the insurer will earn r_A on the investment of equity plus the net return $(r_A k + r_U)$ on underwriting multiplied by the underwriting leverage ratio, s. The insurer has the option of not writing insurance (choosing $s = 0$). In this case, it will be an investment company, investing equity at rate r_A. Writing insurance at a negative underwriting profit will increase return on equity as long as $r_A k > -r_U$. For example, if $k = 1$ and $r_A = .1$, the insurer will make money by writing insurance as long as $r_U > -.1$, i.e., the underwriting loss is less than 10 percent of premiums.

Lines with longer payout tails will have larger funds generating factors (higher k) because the insurer holds policyholder funds for a longer period before claims are paid. Such lines will support a larger underwriting loss than lines with shorter payout tails. For example, if $k = 2$ and $r_A = .1$, the insurer will make money by underwriting insurance as long as the underwriting loss is less than 20 percent. A larger underwriting loss also can be supported when interest rates (r_A) are relatively high. The model thus provides an explanation for the inverse relationship between the length of the payout tail and underwriting profits and well as the inverse relationship between underwriting profits and interest rates.

The Insurance CAPM

Even though insurers can increase the return on equity by writing at an underwriting loss up to the point where $r_A k = -r_U$, this is not necessarily the equilibrium rate of underwriting return in a competitive market. To determine the equilibrium return it is necessary to introduce an asset pricing model. The earliest financial models of insurance were based on the capital asset pricing model (CAPM) (see Biger and Kahane (1978) and Fairley (1979)). According to the CAPM, the equilibrium rate of return on any asset is:

$$r_i = r_f + \beta_i(r_m - r_f) \tag{41}$$

where r_i = the expected return on asset i,
r_f = the risk-free rate of interest,
r_m = the expected return on the market portfolio, and
β_i = the systematic risk coefficient or beta of asset i = $\mathrm{Cov}(r_i, r_m)/\mathrm{Var}(r_m)$.

The CAPM indicates that the asset will earn the risk free rate of interest (usually interpreted as the rate of return on 3 month treasury bills) plus a risk premium. The risk premium is proportional to $(r_m - r_f)$, which is the risk premium an investor could earn by investing in a portfolio with the same risk as the market as a whole. The proportionality factor is the asset's beta, a measure of covariability of the asset return with the market return.

The CAPM implies that investors will be rewarded for bearing systematic or beta risk but not for taking unsystematic risk, i.e., risk that is uncorrelated with the market return. The reason is that this risk can be diversified away by holding a properly structured asset portfolio. The market will not reward an investor for risk that can be reduced or eliminated through diversification. Under the assumptions of the CAPM, investors are assumed to hold efficient portfolios. An efficient portfolio has the highest possible return for its risk level or, equivalently, the lowest possible risk for a given expected return.

Equations (39) and (41) can be used to derive the equilibrium rate of underwriting return. If the CAPM holds, both the rate of return on the insurer's equity, r_E, and the return on its asset portfolio, r_A, must satisfy the CAPM pricing equation. Substituting the appropriate pricing relationships into (39), one can solve for the equilibrium underwriting return.

To express the required underwriting return in its simplest form, it is necessary to know the beta of the insurer's equity. This is obtained by multiplying equation (39) by the market return r_m and taking the covariance of the resulting relationship:

$$Cov(r_E, r_m) = (ks+1)Cov(r_A, r_m) + s\ Cov(r_U, r_m) \tag{42}$$

Dividing (42) by $Var(r_m)$, one obtains: $\beta_E = (ks+1)\beta_A + s\beta_U$, where β_U = the underwriting beta = $Cov(r_U, r_m)/Var(r_m)$ and the other betas are defined similarly.

To complete the derivation of the insurance CAPM, substitute β_E for β_i in (41) to obtain the CAPM rate of return on insurer equity (r_E*). Equate the CAPM return with the return given in the simple financial model (39) and solve the resulting expression for r_U. This gives the equation sometimes called the *insurance CAPM:*

$$r_U = -kr_f + \beta_U(r_m - r_f) \tag{43}$$

The equation is the equilibrium rate of underwriting return under the CAPM.

The first term in (43), $-k\ r_f$, represents an interest credit for the use of policyholder funds. Policyholders pay premiums to the company at policy inception and claims are paid, on the average, k periods later. The policyholder is credited at the risk free rate for interest earned by the insurer during this holding period. The second component of r_U is the insurer's reward for risk-bearing. This reward is the underwriting beta multiplied by the market risk premium. Thus, if underwriting profits are positively correlated with the market, the insurer will earn a positive risk loading and the underwriting return will be greater than $-k\ r_f$.

The model treats insurance policies as analogous to the debt instruments issued by non-financial corporations. The insurer borrows funds from policyholders, invests the funds at r_A and pays claims (retires the debt) k periods later. The debt is riskless in the sense that insurers are assumed not to default on their policy obligations.

Because the CAPM rewards the insurer only for bearing systematic risk, the insurer receives no risk loading if insurance risk is purely unsystematic or diversifiable. If insurance risk is negatively correlated with the market, the insurer pays the policyholder a risk premium in addition to the riskless interest credit.

The insurance CAPM yields important insights into the operation of insurance markets. The model focuses attention on the essentially financial nature of the insurance transaction and forces one to identify the specific types of risk that will receive market rewards and the precise relationships that will hold in equilibrium.

In spite of its importance as a conceptual tool, the insurance CAPM has several limitations that must be addressed in moving from its simple conceptual world into the real-world. One serious problem is the use of the funds generating or k factor to represent the payout tail. It is well known in finance that discounted present value methods should be used when valuing cash flows that occur at different periods of time. The k factor represents only a crude approximation of the discounting process that can lead to serious errors when estimating premiums (Myers and Cohn (1987)). Since discount formulas are so well known and widely available, there is really no excuse for ignoring them and using the k factor.

Researchers who resort to the use of k factors at the present time and in the future are retarding the development of the field of insurance economics.

A second problem is the assumption of no bankruptcy. Far from being default-free, insurance debt actually carries a significant probability of default. It should be priced using a risky-debt model rather than a riskless debt model like the CAPM. A third problem is that interest rate risk plays a significant role in the management of insurance pools. The constant risk-free rate inherent in the CAPM thus oversimplifies an important element of risk.

As a practical matter, the use of underwriting betas can lead to inaccuracies because accurate estimates of this type of beta are very difficult to obtain (see Cummins and Harrington (1985)). One reason for this is that reported insurance underwriting returns often are distorted due to reserving errors and other accounting idiosyncracies. Consequently, the estimation techniques and refinements that have been developed to calculate market equity betas are of little help in obtaining underwriting betas.

Discrete Time Discounted Cash Flow Models

A fundamental principle of finance is that the value of any asset is equal to the present value of its cash flows. Since insurance contracts involve cash flows occurring at different periods of time, discounted cash flow methods provide a natural approach to insurance pricing. Although discounted cash flow principles have long formed the basis for life actuarial science, these principles have only recently come into general use in property-liability insurance. Both discrete and continuous time discounting models have been applied to insurance problems. The former are discussed in this section.

A simple two-period model can be used to illustrate some important concepts in discounted cash flow analysis (see Kraus and Ross (1982)). Consider a policy that promises to pay losses that have a value of L_0 at the current price level. The premium is paid at the beginning of the year, and the loss payment is made at the end of the year. The interest rate is r, the general (economy-wide) inflation rate is i, and the insurance inflation rate is π.

Assume that the Fisher hypothesis holds, i.e., $(1+r) = (1+i)(1+r_r)$, where r_r is the real rate of interest. The real rate can be interpreted as the cost of funds in an economy with zero inflation. The discounted cash flow premium is given by:

$$P = \frac{L_0(1+\pi)}{(1+i)(1+r_r)} \tag{44}$$

If insurance inflation equals general inflation, the premium is:

$$p = \frac{L_0}{1+r_r} \tag{45}$$

Thus, if $i = \pi$, the premium can be obtained by discounting the value of the loss obligation in current prices at the real rate. In this case, it is not necessary to forecast loss inflation. However, if insurance and general inflation are not equal, then equation (44) should be used and forecasts of loss inflation are required.[21]

An important insight from the two-period model is the relationship between anticipated inflation, interest rates, and the premium. In addition, since the expected loss value at the beginning of the next period is $L_0(1 + \pi)$, premiums will increase over time even if insurance and general inflation are equal.[22]

The model also illustrates an important relationship between interest rates and reported underwriting profits. To show the simplest form of this relationship, assume that anticipated insurance and general inflation are equal and that actual inflation exactly equals anticipated inflation.[23] Also assume that insurers report incurred losses at full value (i.e., losses are undiscounted and reflect an estimate of inflation during the payout period). Then the underwriting profit is:

$$r_U = \frac{P-L}{P} = \frac{\frac{L_o}{1+r_r} - L_0(1+i)}{\frac{L_0}{1+r_r}} = \frac{L_0 - L_0(1+i)(1+r_r)}{L_0} = -r \tag{46}$$

Thus, reported underwriting profits are expected to be highly correlated with nominal interest rates, a result borne out by numerous empirical studies (e.g., Cummins and Harrington (1985).

Discounted cash flow models that are used in practice are inevitably more complicated that the simple model of equation (44). The best of the current practical models are those developed by Myers and Cohn (1987) and by the National Council on Compensation Insurance (NCCI). The reader is referred to Cummins (1990a) for the details.

Option Pricing Models

Modern option pricing models (OPM) provide valuable intuition about numerous insurance problems. The simplest type of option is a contingent claim which gives owner the right to buy or sell an asset on or before some expiration date at a stated price (the exercise or striking price). Options are derivative securities in the sense that their value is not intrinsic but is derived

---

[21] The discussion pertains to anticipated inflation rates. In actual inflation differs from anticipated inflation, the insurer will incur an underwriting loss that is larger or smaller than expected.

[22] In computing the premium at time 1 (end of the current period), the numerator of (44) will be $L_0(1 + \pi)^2$.

[23] Relaxing these assumptions has no qualitative effect on the point being demonstrated.

from the value of an underlying asset. This discussion focuses on so-called European options, which can be exercised only at the exercise date and not before. A European call option gives the owner the right to buy a given asset at the exercise price (strike price) on the exercise date. At that time, the value of the option is:

$$C(A;\tau,K) = Max[A-K,0] \tag{47}$$

where $C(A;\tau,K)$ = the value of the call option on an asset with value A at time τ away from the expiration date, and
K = the exercise price.

A European put option gives the holder the right to sell the underlying asset at the exercise price on the exercise date. At that time, the put option has the following value:

$$P(A;\tau,K) = Max[K-A,0] \tag{48}$$

where $P(A;\tau,K)$ = the value of the put with exercise price K on an asset with value A, τ periods from expiration. An important relationship is the put-call parity equation:

$$Parity\colon A = [Ke^{-r\tau} - P(A;\tau,K)] + C\,(A;\tau,K) \tag{49}$$

The option pricing model has been applied to insurance pricing by several authors including Smith (1977), Brennan and Schwartz (1979), and Doherty and Garven (1986), and to insurance guaranty fund premia by Cummins (1988). Both Cummins and Doherty-Garven use a more general form of the options relationship in which both assets (A) and the striking price (K) are random. Doherty and Garven (1986) use discrete-time, risk-neutral valuation theory (e.g., Brennan (1979)), while Cummins (1988) uses the continuous time theory underlying the Black-Scholes option pricing formula. For simple cases, these two approaches give equivalent results, but for more complicated problems the results are not identical. This paper focuses on the continuous time approach.

In the continuous time approach, assets and liabilities are defined as diffusion processes:

$$dL = \pi\, L\, dt + \sigma_L L\, dz_L \tag{50a}$$

$$dA = \mu\, A\, dt + \sigma_A A\, dz_A \tag{50b}$$

where dz_A, dz_L = (possibly correlated) standard Brownian motion processes for assets and liabilities, respectively.

Brownian motion is a type of stochastic process that has found wide application in finance.[24] Let z(t), t $\geq$ 0, be the value of a stochastic process at time t. The process z(t) is Brownian motion if it satisfies several mathematical conditions. Two of the most important are the following (Karlin and Taylor

---

[24]The term *Weiner process* is also used for this type of process.

(1975)): (1) Every increment of the process, $z(t+\tau)-z(t)$, is normally distributed with mean $\mu\tau$ and variance $\sigma^2\tau$.[25] Thus, both the mean and the variance depend upon the interval length, τ. (2) Non-overlapping increments are statisticly independent. ***Standard Brownian motion*** is a Brownian process with $\mu = 0$ and $\sigma^2 = 1$.

If a process S(t) follows Brownian motion with parameters μ and σ^2, then any change in the process, e.g., $S(t)-S(t_0)$ is normally distributed with mean $\mu(t-t_0)$ and variance $\sigma^2(t-t_0)$. For example, suppose S is an asset price, that the price at the beginning of the period $S(t_0) = 1$, and that $\mu = .08$ and $\sigma = .06$. If $t-t_0 = .75$ (e.g., three-fourths of one year, where μ is defined as the annual expected rate of change), then $S(t)-S(t_0)$ is normal with expected value .06 (.75μ), variance .0027 (.75 σ^2), and standard deviation .05196. So the expected value at t, $E[S(t)] = S(t_0)+\mu\, t = 1.06$, but the actual value could be greater or less with the difference depending upon a random draw from a normal distribution. Somewhat more formally, we can write: $dS = \mu\, dt + \sigma\, dz$, where z is standard Brownian motion. If the z term were not present, S would increase deterministically following a straight line with slope μ. The presence of the stochastic term (σ dz) means that S will fluctuate from the trend line by random shocks.

The reader will undoubtedly have noticed a difference between the process S(t) and the asset and liability processes defined in equations (50a) and (50b). Most finance applications use a generalization of Brownian motion known as geometric Brownian motion, where it is the rate of change dS/S and not dS itself that is Brownian, i.e., $dS = S\,\mu\, dt + S\,\sigma\, dz$. This is the form of the processes (50a) and (50b). Geometric Brownian motion can be written as: $S(t) = e^{z(t)}$, where z(t) is a Brownian motion process. Rewriting, we have $z(t) = \ln[S(t)]$, so that increments of the natural logarithm of S(t) are normally distributed, implying that increments in S(t) are lognormally distributed. The lognormal has been found to be a better model for asset prices than the normal. The lognormal property implies that the end of period values of assets and liabilities have the following probability distributions:

$$g(L) = \frac{1}{L\sigma_L\sqrt{\tau}\,\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\ln(L/L_0)-(\pi-\frac{\sigma_L^2}{2})\tau}{\sigma_L\sqrt{\tau}}\right)^2} = \Lambda[L;\ln(L_0)+(\pi-\frac{\sigma_L^2}{2})\tau,\ \sigma_L\sqrt{\tau}] \quad (51)$$

$$f(A) = \frac{1}{A\sigma_A\sqrt{\tau}\,\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{\ln(A/A_0)-(\mu-\frac{\sigma_A^2}{2})\tau}{\sigma_L\sqrt{\tau}}\right)^2} = \Lambda[A;\ln(A_0)+(\mu-\frac{\sigma_A^2}{2})\tau,\ \sigma_A\sqrt{\tau}] \quad (52)$$

where $\Lambda[x;\ \alpha,\ \beta]$ = the lognormal distribution of random variable x with location parameter α and risk parameter β.

---

[25] This means that as long as we know z(t), then no value of z(s), s < t, has any effect on our knowledge of the probability distribution of the increment $z(t+\tau)-z(t)$. Thus, Brownian motion is a Markov process.

An important result using continuous time modelling is the option model of the insurance firm with stochastic assets and liabilities. A derivation of this model appears in Cummins (1988); the results are summarized below: Assume that an insurer has initial (time 0) equity E_0. It will write insurance policies with losses valued at L_0 at policy inception for which a premium of G is collected. The policies expire and losses are paid at the end of one period (e.g., one year). Between time 0 and time 1 (policy expiration), the asset and liability processes follow geometric Brownian motion (equations (50a) and (50b)). Specifically, losses grow at a (constant) rate, and are impacted by a random shock term while assets grow at constant rate plus a random shock term. To simplify the notation, insurance liabilities are assumed to have no systematic risk.

At the end of the period, when the liability payments are due, the insurer has an option: it can pay off the liabilities or it can default and turn the company over to the policyholders. It will pay off the liabilities as long as a positive amount of money remains after the liabilities are paid. It will default if the company's assets do not cover the liabilities. Thus, the value of the shareholders' claim on the firm at time 1 (policy expiration) is Max[A – L,0].

This, of course, has the same form as a payoff from a call option, so the shareholders' claim can be viewed as a call option on the firm's assets (A) with exercise price L.

The policyholders' claim at time 1 is the direct analogue of the claim of the bondholders in a levered corporation, i.e., Min[L,A] = L – Max[L – A,0]. The policyholders' claim is thus equal to the value of liabilities less the value of a put option on the firm's assets with striking price L. The put option is often called the insolvency put because it is exercised only if the firm is insolvent.

The values of the relevant calls and puts at the policy expiration date are given by the "Max" statements defined in the preceding paragraphs. Prior to expiration, these can be valued by Black-Scholes option formulas, modified to allow for stochastic liabilities (stochastic exercise price) as well as stochastic assets. Furthermore, the value of assets at the policy issue date is the sum of premiums plus equity (G + E). Because equityholders will not put money into an insurance company if value is immediately lost by doing so and because policyholders will not (in a competitive market) pay more than the fair market value for insurance, the fair premium is obtained by solving (53) for G:

$$C[G+E;\tau,L] = E \tag{53}$$

An equivalent expression from the policyholder perspective is:

$$G = e^{-(r-\pi)\tau}L - P[G+E;\tau,L] \tag{54}$$

In these equations, E, G, and L are all valued τ periods before the expiration of the options. Adding (53) and (54) yields the put-call parity formula. In (54), notice that losses are discounted to the present at discount rate $(r - \pi)$. Of course, if expected economy-wide inflation is equal to insurance inflation, $r - \pi$ = the real rate, providing an analogy with the simple single period certainty model discussed above (equation (45)). However, π could be greater than r, leading to a riskless present value of liabilities greater than L. The formula for the call option in (53) is:

$$C(G+E;\tau,L) = (G+E)\int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx - Le^{-r^*\tau}\int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \tag{55}$$

where $d_1 = \ln((G+E)/(Le^{-r^*t}))/\sigma\sqrt{\tau} + .5\sigma\sqrt{\tau}$,
$d_2 = d_1 - \sigma\sqrt{\tau}$,
$r^* = r - \pi$,
π = instantaneous rate of inflation of insurance claims,
$\sigma^2 = \sigma_A^2 + \sigma_L^2 - 2\sigma_A\sigma_L\rho_{AL}$, and
ρ_{AL} = instantaneous correlation coefficient between dz_A and dz_L.

Using (55), equation (53) can be solved numerically to obtain the fair premium.

There are several interesting points to notice about the option model of insurance pricing. (1) The discount rate is the risk free rate (r) minus the insurance inflation rate. This rate can be negative, leading to higher premiums if insurance inflation is higher than general inflation. (2) The optioned variable is actually the asset-to-liability ratio, A/L = (G+E)/L. Thus, the option is denominated in liabilities and the homogeneity property of options is used to write equation (55) in terms of nominal values rather than the A/L ratio. (3) By using continuous time mathematics, an exact lognormal result is obtained that reflects stochastic assets and stochastic liabilities. By contrast, the lognormal formula developed by Doherty and Garven using discrete time modelling is only approximate because sums of lognormal variables are not lognormal. (4) Positive correlation between asset and liability returns reduces the risk parameter and increases the premium. Such positive correlation means that the insurer has available a natural hedge so the insurance policies are safer and hence sell for more.

Although the European option formula provides significant insights into insurance pricing, it is not really appropriate for most real-world insurance problems. Few insurance problems satisfy the rigid design features of this model, i.e., no payment until a fixed expiration date when all claims are paid. Unfortunately, some authors have attempted to force insurance problems into the European option framework. The more appropriate, but more difficult, approach is to adapt the model to fit the problem instead of forcing the problem to fit the model. Adapting the model usually involves non-trivial extensions of the underlying mathematics. However, it is well worth the effort because even complicated insurance contracts have option-like characteristics. The finance literature contains a wealth of more advanced contingent claim models that have never been applied to insurance (see, for example, Ingersoll (1987)).

An example of a situation where the European option model probably is appropriate without much modification is the calculation of guaranty fund premia (see Cummins (1988)). The guaranty fund promises to pay policyholders if the company defaults. The option is on the entire company and can be viewed as having a fixed term. For example, assume that the guaranty is for one period. Then the value of the guaranty at the end of the period, when the company is audited, is Max[L-A, 0], i.e., the guaranty fund pays only if liabilities exceed assets at the audit date. Thus, the value of the guaranty can be computed as the value of a put option.[26]

Other insurance problems can be formulated in terms of option theory. One example is stop-loss reinsurance. Consider the following stop-loss arrangement on the total losses of an insurance company, L, during some period of time such as one year. At the end of the specified period, if $L > M$, where M is the point of attachment, the reinsurer will pay proportion α $(0 < \alpha < 1)$ of losses up to an upper limit R where the primary insurer again becomes responsible for 100 percent of losses.[27] Thus, the reinsurer's obligation can be stated as follows:

$$L_R = 0, \qquad L \leq M$$

$$L_R = \alpha(L-M), \; M < L < R$$

$$L_R = \alpha(R-M), \; L \geq R$$

where L_R = the loss of the reinsurer. The reinsurer's end-of-period loss obligation can also be written as follows: $\alpha(\text{Max}[L-M, 0] - \text{Max}[L-R, 0])$. Thus, the obligation, which equals the cost of the reinsurance contract, is α times the difference between two call options on the random variable L, with striking prices M and R, respectively.

In using options models for insurance problems, one must keep in mind that most insurance variables are non-price variables that are not subject to continuous trading. The lack of trading can be circumvented in some instances by using the discrete-time, risk-neutral valuation approach as in Doherty and Garven (1986). In other instances, the accuracy lost by assuming continuous trading may not be very great. However, modifications to the standard option formulas may be needed to handle these and other problems (see Ingersoll (1987)).

---

[26] Of course, in practice, states do not charge insurers risk-based premiums. Guaranty funds are not prefunded, except in New York, and are based on flat assessments proportional to the size of each insurer. As pointed out in Cummins (1988), flat rather than risk-based premiums create a type of "lemons" problem and may cause insurers to adopt high-risk strategies increasing the number of insolvencies. Adopting risk-based premiums would solve this problem. Options models would be the correct way to price risk-based premiums because the guaranty fund coverage is consistent with the options framework. Other types of insurance contracts, e.g., long-tail liability policies, do not fit into the standard options framework. The confluent hypergeometric model developed in Cummins (1988) is consistent with long-tail insurance policies and could be adapted for this pricing problem.

[27] The stop-loss contract could be stated equivalently in terms of loss ratios rather than losses.

Asset-Liability Management

Property-liability insurers are financial institutions, holding asset portfolios to back up short and intermediate-term liabilities. Both the asset and liability values are subject to shocks due to changes in interest rates. Liabilities are also subject to inflation risk.

Modern financial theory provides methodologies for reducing or eliminating risk due to interest rate fluctuations for firms that hold portfolios of fixed income securities and/or fixed income liabilities. Collectively, these methodologies fall under the heading of asset-liability management techniques. Such techniques are only beginning to be applied in property-liability insurance. The following is an overview of some of the basic concepts.

The first step in applying an asset-liability management technique is to decide what to manage. Asset-liability management can be applied to a number of different target accounts such as the market value of equity, the market leverage ratio (assets divided by equity), market value of net income, book value of net income, etc. The market value of the equity and the market leverage ratio would seem to make the most sense in property-liability insurance. Insurers are in business to provide financial guarantees. Consequently, protection of equity against interest rate risk would seem to be a high priority. In addition, buyers and regulators are likely to gauge the financial stability of an insurer by considering financial ratios that are highly correlated with the market asset-to-equity ratio. These two ratios are the focus of the discussion to follow. Management of other target accounts is discussed in Toevs and Haney (1986).

The problem to be solved in asset-liability management is the reduction or elimination of interest rate risk as it affects the target accounts. Interest rate risk exists for institutions that hold portfolios of assets and portfolios of liabilities, either of which is sensitive to changes in interest rates. For example, an increase in interest rates will lead to a decline in the market value of a portfolio of bonds. If liabilities also have the characteristics of a fixed income portfolio, they too will decline in market value as interest rates increase. The net effect depends upon the relative sensitivity of assets and liabilities to the interest rate changes.

An important measure of interest rate sensitivity is the Macaulay duration (Macaulay (1938)). The duration can be understood intuitively as the weighted average time to maturity of an asset where the weights are the market values of the asset's cash flows. The formulas for the value of an asset and the asset's duration are:

$$P = \sum_{i=1}^{N} \frac{C_i}{R^i} \tag{56}$$

$$D = -\frac{dP}{dR}\frac{R}{P} = \frac{\sum_{i=1}^{N} \frac{iC_i}{R^i}}{P} \tag{57}$$

where P = price of the asset,
C_i = the amount of the ith cash flow,
N = the number of cash flows,
R = the discount factor, R = 1 + r, and
D = the duration of the asset.

The duration D is obtained by differentiating the equation for P with respect to the discount factor R. As (57) shows, the duration is -1 times the elasticity of the asset's price with respect to the discount factor.

The discrete time version of the duration formula can be used to estimate the asset price change resulting from a change in the yield rate, r.[28] This is the following formula:

$$\Delta P = -D \frac{\Delta R}{R} P \tag{58}$$

For example, consider a four-year semi-annual coupon bond selling at par to yield 9.3 percent. The bond's duration is 3.43 years. If interest rates go up by 1 percent to 10.3 percent, equation (58) implies that the price change will be −3.1 percent so that the price of the bond falls to $969.

A convenient feature of the duration measure is that the duration of a portfolio is merely the weighted average of the durations of the individual assets comprising the portfolio with the weights equal to the market value proportion that each asset represents in the portfolio. This is expressed in equation (59):

$$D = W_1 D_1 + W_2 D_2 + \ . \ . \ . \ + W_k D_k \tag{59}$$

where D = the duration of the portfolio,
D_i = the duration of security i,
P_i = the price (market value) of the ith asset,
P = $P_1 + \ . \ . \ . \ + P_k$ = the market value of the portfolio, and
$W_i = P_i / P$.

Equation (59) is quite general and applies to portfolios consisting of both long and short positions in various securities. The primary constraint is that $\Sigma_i W_i = 1$.

Equation (59) can be used to weight together the durations of the individual securities and liability categories comprising an insurer's asset and liability portfolios to obtain the durations of assets and liabilities. These durations also can be weighted together using (59) to obtain the duration of the insurer's equity. The key insight is that liabilities can be considered an asset that has been sold short and hence enters the duration equation with a negative sign.

[28]The duration formula is obtained using calculus techniques. Consequently, it is strictly accurate only for very small changes in interest rates and prices. However, the discrete time version has been shown to give rather good approximations to the true price changes, except for relatively large changes in interest rates.

Equity thus represents a *net bond* reflecting the net flows to the equity holders from the firm's assets and liabilities.

The formula for the duration of equity is the following:

$$D_E = \frac{A}{E} D_A - \frac{L}{E} D_L \tag{60}$$

where D_E, D_A, and D_L = the durations of equity, assets, and liabilities, and
E, A, L = the market values of equity, assets, and liabilities.

For example, suppose that the asset to equity ratio is 4.0, so that the liability to equity ratio is 3.0. Also assume that the duration of assets is 4.5 and the duration of liabilities is 2.5. The duration of equity will be 10.5 years. A duration of this magnitude exposes the insurer's equity to considerable interest rate risk. If the discount factor increases by 1 percent, for example, the insurer's equity value will decrease by more than 10 percent.

A duration measure also can be obtained for the market leverage ratio (A/E). This measure is the following:

$$D_{A/E} = -\frac{d(A/E)}{dR} \frac{R}{A/E} = D_E - D_A \tag{61}$$

Another convenient way to write (61) is as follows:

$$D_{A/E} = \frac{L}{E} [D_A - D_L] \tag{62}$$

A common goal in asset-liability management is immunization. This is defined as the protection of equity against interest rate risk and is usually achieved when $D_E = 0$, i.e., when the durations of assets and liabilities are equal. If the portfolio is immunized, changes in interest rates have no effect on net worth. An initially immunized position must be periodically readjusted to take into account duration drift, which involves the unequal shifting of asset and liability durations over time.

Considering equations (60) and (61), it is apparent that the insurer will not be able to protect both equity and the leverage ratio from interest rate risk simultaneously. If the duration of equity is 0, the duration of the leverage ratio cannot also be zero except in unrealistic cases such as having a asset portfolio with zero duration.[29] Thus, the insurer must choose which of the target accounts to manage or arrive at an optimal target-account duration vector. Ultimately, both types of duration will be endogenous to the market, i.e., companies which choose the right combination of assets and liability maturities will be the ones that survive and prosper in a competitive market.

As with many other modeling procedures, the standard duration management techniques do not quite apply in property-liability insurance. Duration models are usually applied to institutions that have liabilities

[29] This would amount to having all assets in the form of overnight deposits or a similar very short term investment.

denominated in nominal dollars. Examples include banks, savings and loans, and life insurance companies where payouts (certificates of deposit, life insurance face values) are in specified dollar amounts that do not vary with inflation. As shown above, the liability accounts of property-liability insurers do vary with inflation. This has a potentially important impact on the duration of liabilities and hence on the durations of equity and leverage.

Consider the case where insurance inflation is the same as general inflation. Also assume that the real rate of interest is constant. In this case the market price of liabilities is given by:

$$P_L = \sum_{i=1}^{N} \frac{L_i}{(1+r_r)^t} \tag{63}$$

where P_L = market value of liabilities,
L_i = the expected liability payout at time i, valued in time 0 prices, and
r_r = the real rate, assumed constant.

Assuming that the real rate is constant is equivalent to saying that all changes in nominal interest rates are the result of inflation. Since inflation does not appear anywhere in equation (63), having been canceled out through the discounting process, liabilities are not sensitive at all to anticipated inflation. Thus, the duration of liabilities is zero. Equity cannot be immunized under these conditions except by holding very short term securities, and the duration of the leverage ratio is $(L/E)D_A$.

A more realistic result, especially in recent years, is that insurance inflation is correlated with and in excess of general inflation. In this case, the sign of liability duration may be opposite to the sign that would pertain if liabilities were not related to inflation, i.e., it may be negative instead of positive. Thus, the liability duration may add to rather than reducing asset duration. The duration of equity and the duration of the leverage ratio would both be higher under these conditions than they would be if the usual relationships held. This hypersensitivity to interest rate (inflation) risk may help to explain the coverage shortages during recent periods of high insurance inflation.

In order for duration measures to be more useful in insurance markets, it will be necessary for insurers to begin to report the true market values of their asset portfolios. The liability portfolios can be valued with reasonable accuracy using available techniques. However, valuing the assets requires detailed knowledge of the individual securities making up insurer portfolios, and is a task beyond the resources of most researchers. The NAIC can help, by requiring insurers to report the market values of their bond portfolios. This information would be of great value to investors, regulators, and insurance buyers and would introduce an added element of market discipline and rationality into insurance markets.

Summary And Conclusions

Actuaries and statisticians have developed highly sophisticated mathematical models of insurance pricing and the insurance firm. The individual and collective risk theory models provide powerful tools for analyzing insurance risk pools. Significant progress has been made in solving some of the computational problems that have long plagued the risk theory field. With more powerful computers and more advanced algorithms, it has become possible to estimate parameters for a much wider class of probability of loss distributions and to compound the estimated distributions with frequency to obtain accurate estimates of the probability distributions of total claims. A complementary development has been the emergence of reserving models that can be linked with the collective risk model to provide a more accurate picture of the loss payout process. Contrary to traditional actuarial practice in the United States, the most advanced reserving models are based on paid rather than incurred loss data.

Although significant progress has been made in modelling insurance pools, the most sophisticated existing models remain statistical rather than economic. The well-trained analyst can calculate means, variances, and higher moments of claim costs as well as ruin probabilities, reserve runoffs, and other statistics. However, the analyst typically cannot provide information on what the price, ruin probability, or reinsurance retention should be. Furthermore, the economic objectives of the insurer, apart from avoiding ruin with some "high" probability, remain more or less undefined in the statistical theory of insurance. This field could gain significantly by integrating with financial theory, which provides some of the missing pieces of the puzzle.

In contrast to statistical models, financial models integrate insurance variables into an economic context. Many of the resulting models are consistent with financial market equilibrium, or, minimally, with the avoidance of arbitrage. In order to achieve this objective, some mathematical simplification usually is required. As a result, financial models do not do as good a job as the statistical models in representing the underlying stochastic processes of insurance. Many of the financial models are based on the hypothesis that insurance loss processes can be approximated by lognormal distributions. While lognormal distributions are acceptable in some instances, they have been shown to underrepresent the tails of many probability of loss distributions observed in practice. Thus, many actual insurance claim processes are more risky than those reflected in the finance literature. An important step towards the integration of insurance financial and statistical theory would be to develop models that incorporate more realistic insurance claim distributions into a financial context.

Other areas in which the integration of statistical and financial models would be valuable are the following: (1) The development of asset/liability management models that take into account more sophisticated models of the reserve runoff. (2) Financial modeling of reinsurance using option pricing theory. Viable models would need to reflect the fact that insurance claims are non-traded assets. They should also incorporate probability distributions other than the lognormal. (3) Development of multi-period option pricing models for long-tail insurance contracts. Such models have the potential to ovide a better representation of the pricing of default risk. Serious researchers should stop attempting to model the claims runoff process using funds generating or "k" factors. This is a regressive approach that does not advance the field. (4) Adaptation of pricing and asset/liability management models to incorporate stochastic interest rates. And (5) the endogenization of surplus. This latter project is particularly difficult and important. Ultimately, the degree of safety (level of ruin probabilities) in insurance markets is endogenous, driven by the demand for security by insurance buyers and the cost of capital in financial markets. The endogenization of surplus would potentially involve the integration of insurance demand theory with the financial and statistical models of the insurance firm.

REFERENCES

Beard, R. E., T. Pentikainen, and E. Pesonen, 1984, Risk Theory, 3d ed. (New York: Chapman and Hall).

Biger, Nihum and Yehuda Kahane, 1978, Risk Considerations In Insurance Ratemaking, *Journal of Risk and Insurance,* 45: 121-132.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy,* 81: 637-654.

Borch, Karl, 1974, *The Mathematical Theory of Insurance* (Lexington, MA: Lexington Books).

Bowers, Newton L., Jr., et al., 1986, *Actuarial Mathematics* (Chicago: The Society of Actuaries).

Brealey, Richard A. and Stewart C. Myers, 1988, *Principles of Corporate Finance,* 3rd ed. (New York: McGraw-Hill Book Co.).

Brennan, Michael J., 1979, The Pricing of Contingent Claims In Discrete Time Models, *Journal of Finance* 24: 53-68.

Brennan, Michael J. and Eduardo S. Schwartz, 1979, *Pricing and Investment Strategies for Guaranteed Equity Linked Life Insurance* (Philadelphia, PA: S.S. Huebner Foundation).

Brockett, Patrick L., 1983, On the Misuse of the Central Limit Theorem in Some Risk Calculations, *Journal of Risk and Insurance* 50: 727-731.

Buhlmann, Hans, 1970, *Mathematical Methods in Risk Theory* (New York: Springer-Verlag).

Casualty Actuarial Society, 1990, *Foundations of Casualty Actuarial Science* (New York: The Society).

Cramer, Harald, 1946, Mathematical Methods of Statistics (Princeton, N.J.: Princeton University Press).

Cummins, J. David, Georges Dionne, James McDonald, and Michael Pritchett, 1990, Applications of the GB2 Family of Probability Distributions In Collective Risk Theory," *Insurance: Mathematics and Economics.* 9:257-272.

Cummins, J. David and Lena Chang, 1983, An Analysis of the New Jersey Formula for Including Investment Income In Property-liability Insurance Ratemaking, *Journal of Insurance Regulation* 1: 555-573.

Cummins, J. David, et al., 1984, *Risk Classification In Life Insurance* (Norwell, MA: Kluwer Academic Publishers).

Cummins, J. David and Scott E. Harrington, 1985, Property-Liability Insurance Rate Regulation: Estimation of Underwriting Betas Using Quarterly Profit Data, *Journal of Risk and Insurance* 52: 16–43.

______. 1987, *Fair Rate of Return In Property-Liability Insurance* (Norwell, MA: Kluwer Academic Publishers).

Cummins, J. David and David J. Nye, 1980, The Stochastic Characteristics of Property-Liability Insurance Profits, *Journal of Risk and Insurance* 47: 61–80.

Cummins, J. David and Laurel L. Wiltbank, 1983, Estimating the Total Claims Distribution Using Multivariate Frequency and Severity Distributions, *Journal of Risk and Insurance* 50 : 377–403.

______. 1984, A Multivariate Model of the Total Claims Process, *Astin Bulletin* 14: 45–52.

Cummins, J. David, 1974, Insurer's Risk: A Restatement, *Journal of Risk and Insurance* 41 : 147–157.

Cummins, J. David, 1988a, Risk-Based Premiums for Insurance Guaranty Funds, *Journal of Finance* 43: 823–839.

Cummins, J. David, 1988b, Capital Structure and Fair Profits In Property-Liability Insurance, working paper, Center for Research on Risk and Insurance, University of Pennsylvania.

Cummins, J. David, 1990a, Multi-Period Discounted Cash Flow Ratemaking Models in Property-Liability Insurance, *Journal of Risk and Insurance* 57: 79–109.

Cummins, J. David, 1990b, Asset Pricing Models and Insurance Ratemaking, *Astin Bulletin* 20: 125–166.

Cummins, J. David, 1991, Financial Pricing of Property-Liability Insurance, forthcoming in Georges Dionne, ed., *Contributions To Insurance Economics* (Norwell, MA: Kluwer Academic Publishers).

D'Arcy, Stephen and Neil A. Doherty, 1988, *Financial Theory of Pricing Property-Liability Insurance Contracts* (Philadelphia: S.S. Huebner Foundation).

D'Arcy, Stephen and James R. Garven, 1990, Property-Liability Insurance Pricing Models: An Empirical Evaluation, *Journal of Risk and Insurance* 57: 391–430. July.

Derrig, Richard A., 1985, The Effect of Federal Income Taxes on Investment Income in Property-Liability Ratemaking, working paper, Automobile Insurers Rating Bureau, Boston, MA.

Doherty, Neil A. and James R. Garven, 1986, Price Regulation in Property-Liability Insurance: A Contingent Claims Approach, *Journal of Finance* 41: 1031–1050.

Fairley, William, 1979, Investment Income and Profit Margins In Property-Liability Insurance: Theory and Empirical Tests, *Bell Journal* 10: 192–210.

Feller, William, 1968, *An Introduction To Modern Probability Theory and Its Applications,* vol. I, 3d ed. (New York: John Wiley & Sons).

Feller, William, 1971, *An Introduction to Probability Theory and Its Applications,* vol. II (New York: John Wiley & Sons).

Ferrari, J. Robert, 1968, A Note on the Basic Relationship of Underwriting, Investments, Leverage, and Exposure to Total Return on Owners' Equity, *Proceedings of the Casualty Actuarial Society* 55: 295-302.

Gerber, Hans, 1982, On the Numerical Evaluation of a Family of Compound Distributions, *Insurance: Mathematics and Economics* 1: 13-18.

Hogg, Robert V. and Allen T. Craig, 1978, *Introduction To Mathematical Statistics,* 4th ed. (New York: Macmillan Publishing Company).

Hogg, Robert V. and Stuart A. Klugman, 1984, *Loss Distributions* (New York: John Wiley & Sons).

Ingersoll, Jonathan E. Jr., 1987, *Theory of Financial Decision Making* (Totowa, NJ: Rowman & Littlefield).

Karlin, Samuel and H.M. Taylor, 1975, *A First Course In Stochastic Processes* (New York: Academic Press).

Kraus, Alan and Stephen Ross, 1982, The Determination of Fair Profits for the Property-Liability Insurance Firm, *Journal of Finance* 33: 1015-1028.

Lemaire, Jean, 1985, *Automobile Insurance: Actuarial Models* (Norwell, MA: Kluwer Academic Publishers).

Macaulay, Frederick R., 1938, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1856* (New York: Columbia University Press).

Mayers, David and Clifford W. Smith, Jr., 1982, On the Corporate Demand for Insurance, *Journal of Business* 55: 281-296.

McDonald, James and Richard Butler, 1987, Some Generalized Mixture Distributions With an Application to Employment Duration, *Review of Economics and Statistics* 69: 232-240.

Myers, Stewart and Richard Cohn, 1987, Insurance Rate Regulation and the Capital Asset Pricing Model, in J. D. Cummins and S. E. Harrington, eds., *Fair Rate of Return In Property-Liability Insurance* (Norwell, MA: Kluwer Academic Publishers).

Panjer, Harry H., 1981, Recursive Evaluation of a Family of Compound Distributions, *Astin Bulletin* 12: 22-26.

Panjer, Harry H. and B.W. Lutek, 1983, Practical Aspects of Stop-Loss Calculations, *Insurance: Mathematics and Economics* 2: 159-177.

Parzen, Emanuel, 1960, *Modern Probability Theory and Its Applications* (New York: John Wiley & Sons).

Paulson, A.S. and N. J. Faris, 1985, A Practical Approach to Measuring the Distribution of Total Annual Claims, in J. David Cummins, ed., *Strategic Planning and Modelling in Property-Liability Insurance* (Norwell, MA: Kluwer Academic Publishers).

Paulson, A.S. and R. Dixit, 1989, Some General Approaches to Computing Total Loss Distributions and the Probability of Ruin, in J. D. Cummins and R. A. Derrig, eds., *Financial Models of Insurance Solvency* (Norwell, MA: Kluwer Academic Publishers).

Rothschild, Michael and Joseph E. Stiglitz, 1976, Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information, *Quarterly Journal of Economics* 90: 629–650.

Samuelson, Paul A., 1963, Risk and Uncertainty: A Fallacy of Large Numbers, *Scientia,* pp. 1–6.

Smith, Clifford W., Jr., 1977, Applications of Option Pricing Analysis, in James L. Bicksler, ed., *Handbook of Financial Economics* (Amsterdam: North-Holland Publishing Company).

Smith, Clifford W., Jr., 1986, On the Convergence of Insurance and Finance Research, *Journal of Risk and Insurance* 53: 693–717.

Taylor, Gregory C., 1977, Separation of Inflation and Other Effects from the Distribution of Non-Life Insurance Claim Delays, *Astin Bulletin* 9: 217–230.

Taylor, Gregory C., 1986, *Claim Reserving In Non-Life Insurance* (New York: North-Holland).

Toevs, Alden L and William C. Haney, 1986, Measuring and Managing Interest Rate Risk: A Guide To Asset/Liability Models Used In Banks and Thrifts, in Robert B. Platt, ed., *Controlling Interest Rate Risk* (New York: John Wiley & Sons).

LINKED CITATIONS

Statistical and Financial Models of Insurance Pricing and the Insurance Firm
J. David Cummins
The Journal of Risk and Insurance, Vol. 58, No. 2. (Jun., 1991), pp. 261-302.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28199106%2958%3A2%3C261%3ASAFMOI%3E2.0.CO%3B2-U

[Footnotes]

[11] **Some Generalized Mixture Distributions with an Application to Unemployment Duration**
James B. McDonald; Richard J. Butler
The Review of Economics and Statistics, Vol. 69, No. 2. (May, 1987), pp. 232-240.
Stable URL:
http://links.jstor.org/sici?sici=0034-6535%28198705%2969%3A2%3C232%3ASGMDWA%3E2.0.CO%3B2-Y

References

Risk Considerations in Insurance Ratemaking
Nahum Biger; Yehuda Kahane
The Journal of Risk and Insurance, Vol. 45, No. 1. (Mar., 1978), pp. 121-132.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28197803%2945%3A1%3C121%3ARCIIR%3E2.0.CO%3B2-R

The Pricing of Options and Corporate Liabilities
Fischer Black; Myron Scholes
The Journal of Political Economy, Vol. 81, No. 3. (May - Jun., 1973), pp. 637-654.
Stable URL:
http://links.jstor.org/sici?sici=0022-3808%28197305%2F06%2981%3A3%3C637%3ATPOOAC%3E2.0.CO%3B2-P

NOTE: *The reference numbering from the original has been maintained in this citation list.*

LINKED CITATIONS

The Pricing of Contingent Claims in Discrete Time Models
M. J. Brennan
The Journal of Finance, Vol. 34, No. 1. (Mar., 1979), pp. 53-68.
Stable URL:
http://links.jstor.org/sici?sici=0022-1082%28197903%2934%3A1%3C53%3ATPOCCI%3E2.0.CO%3B2-Q

On the Misuse of the Central Limit Theorem in Some Risk Calculations
Patrick L. Brockett
The Journal of Risk and Insurance, Vol. 50, No. 4. (Dec., 1983), pp. 727-731.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28198312%2950%3A4%3C727%3AOTMOTC%3E2.0.CO%3B2-R

Property-Liability Insurance Rate Regulation: Estimation of Underwriting Betas Using Quarterly Profit Data
J. David Cummins; Scott Harrington
The Journal of Risk and Insurance, Vol. 52, No. 1. (Mar., 1985), pp. 16-43.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28198503%2952%3A1%3C16%3APIRREO%3E2.0.CO%3B2-9

The Stochastic Characteristics of Property-Liability Insurance Company Underwriting Profits
J. David Cummins; David J. Nye
The Journal of Risk and Insurance, Vol. 47, No. 1. (Mar., 1980), pp. 61-77.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28198003%2947%3A1%3C61%3ATSCOPI%3E2.0.CO%3B2-O

Estimating the Total Claims Distribution Using Multivariate Frequency and Severity Distributions
J. David Cummins; Laurel J. Wiltbank
The Journal of Risk and Insurance, Vol. 50, No. 3. (Sep., 1983), pp. 377-403.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28198309%2950%3A3%3C377%3AETTCDU%3E2.0.CO%3B2-Z

Insurer's Risk: A Restatement
J. David Cummins
The Journal of Risk and Insurance, Vol. 41, No. 1. (Mar., 1974), pp. 147-157.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28197403%2941%3A1%3C147%3AIRAR%3E2.0.CO%3B2-Y

NOTE: *The reference numbering from the original has been maintained in this citation list.*

LINKED CITATIONS

Risk-Based Premiums for Insurance Guaranty Funds
J. David Cummins
The Journal of Finance, Vol. 43, No. 4. (Sep., 1988), pp. 823-839.
Stable URL:
http://links.jstor.org/sici?sici=0022-1082%28198809%2943%3A4%3C823%3ARPFIGF%3E2.0.CO%3B2-7

Multi-Period Discounted Cash Flow Rate-making Models in Property-Liability Insurance
J. David Cummins
The Journal of Risk and Insurance, Vol. 57, No. 1. (Mar., 1990), pp. 79-109.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28199003%2957%3A1%3C79%3AMDCFRM%3E2.0.CO%3B2-0

Property-Liability Insurance Pricing Models: An Empirical Evaluation
Stephen P. D'Arcy; James R. Garven
The Journal of Risk and Insurance, Vol. 57, No. 3. (Sep., 1990), pp. 391-430.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28199009%2957%3A3%3C391%3APIPMAE%3E2.0.CO%3B2-9

Price Regulation in Property-Liability Insurance: A Contingent-Claims Approach
Neil A. Doherty; James R. Garven
The Journal of Finance, Vol. 41, No. 5. (Dec., 1986), pp. 1031-1050.
Stable URL:
http://links.jstor.org/sici?sici=0022-1082%28198612%2941%3A5%3C1031%3APRIPIA%3E2.0.CO%3B2-8

The Determination of Fair Profits for the Property-Liability Insurance Firm
Alan Kraus; Stephen A. Ross
The Journal of Finance, Vol. 37, No. 4. (Sep., 1982), pp. 1015-1028.
Stable URL:
http://links.jstor.org/sici?sici=0022-1082%28198209%2937%3A4%3C1015%3ATDOFPF%3E2.0.CO%3B2-%23

On the Corporate Demand for Insurance
David Mayers; Clifford W. Smith, Jr.
The Journal of Business, Vol. 55, No. 2. (Apr., 1982), pp. 281-296.
Stable URL:
http://links.jstor.org/sici?sici=0021-9398%28198204%2955%3A2%3C281%3AOTCDFI%3E2.0.CO%3B2-W

NOTE: *The reference numbering from the original has been maintained in this citation list.*

LINKED CITATIONS

Some Generalized Mixture Distributions with an Application to Unemployment Duration
James B. McDonald; Richard J. Butler
The Review of Economics and Statistics, Vol. 69, No. 2. (May, 1987), pp. 232-240.
Stable URL:
http://links.jstor.org/sici?sici=0034-6535%28198705%2969%3A2%3C232%3ASGMDWA%3E2.0.CO%3B2-Y

Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information
Michael Rothschild; Joseph Stiglitz
The Quarterly Journal of Economics, Vol. 90, No. 4. (Nov., 1976), pp. 629-649.
Stable URL:
http://links.jstor.org/sici?sici=0033-5533%28197611%2990%3A4%3C629%3AEICIMA%3E2.0.CO%3B2-N

On the Convergence of Insurance and Finance Research
Clifford W. Smith, Jr.
The Journal of Risk and Insurance, Vol. 53, No. 4. (Dec., 1986), pp. 693-717.
Stable URL:
http://links.jstor.org/sici?sici=0022-4367%28198612%2953%3A4%3C693%3AOTCOIA%3E2.0.CO%3B2-Q