Capital Allocation

"Capital Allocation for Insurance Companies"

Stewart C. Myers and James A. Read, Jr.

2001 *Journal of Risk and Insurance*

Motivation

- MR show how option pricing methods can be used to allocate capital across lines of insurance.
- Why is this an interesting question?
  - Capital allocation is an important consideration for modeling the pricing of intermediated risks (e.g., insurance policies and bank loans).
  - Common industry practices (e.g., uniform capital allocation) are logically inconsistent with insurance bankruptcy law which conveys equal priority in bankruptcy.

Some Definitions

- Aggregate liabilities equal the sum of the present value of losses on each line; i.e., $L = \sum_{i=1}^{M} L_i$, where $L_i \equiv PV(L_i)$.

- Aggregate surplus equals the sum of line-by-line surplus contributions, which are proportional to liabilities, so $S = \sum_{i=1}^{M} L_i\, s_i = L\, s$, where $s_i \equiv \dfrac{\partial S}{\partial L_i}$ is the surplus required per dollar of liabilities in Line *i* and $s \equiv S / L$ is the aggregate surplus-to-liability ratio.

- Assets equal the sum of liabilities and surplus:

$$V = \sum_{i=1}^{M} L_i (1 + s_i) = L(1 + s)$$

Some Definitions

- The aggregate surplus ratio is a weighted average of the line-by-line surplus requirements, $s = \sum_{i=1}^{M} x_i\, s_i$, where $x_i \equiv L_i / L$.
- The line-of-business default allocations d_i are defined as the *marginal* contributions to the default value of the company: $d_i \equiv \partial D / \partial L_i$.
- The sum of the products of line-by-line liabilities and marginal default values is equal to the default value for the company: $\sum_{i=1}^{M} L_i\, d_i = D$.

End-of-Period Payoffs

- $E_1 = Max\{0, (V_1 - L_1)\}$; i.e., equity represents a call option on the firm's assets (V_1) with an exercise price equal to aggregate realized losses.

- Alternatively, $E_1 = V_1 - L_1 + D_1$, where $D_1 = Max\{0, (L_1 - V_1)\}$; i.e., equity consists of the sum of an unlimited liability payoff ($V_1 - L_1$) plus the payoff on a "default" option.

- The default option can be dynamically replicated by a combination of positions in assets (V) and liabilities (L); i.e., $$D = \frac{\partial D}{\partial L} L + \frac{\partial D}{\partial V} V.$$

Marginal Default Values – Lognormal Case

- If assets and liabilities are jointly lognormal, then $\frac{\partial D}{\partial L} = N\{z\}$ and $\frac{\partial D}{\partial V} = -N\{z - \sigma\}$, where $z = \ln\left(\frac{L}{V}\right)/\sigma + .5\sigma$ and σ is the standard deviation of the asset-liability ratio.
    - Note that these comparative static results are comparable to the put option comparative statics $\partial P / \partial X > 0$ and $\partial P / \partial S < 0$ from an earlier lecture.
- Thus the value of the option to default $D = f(L, V, \sigma)$.

Marginal Default Values – Lognormal Case

- Computation of σ requires the computation of 1) the variance (σ_L^2) of "log" losses, 2) the variance (σ_V^2) of "log" assets, and the covariance between "log" losses and "log" assets (σ_{LV}).
  - Thus, $\sigma = \sqrt{\sigma_L^2 + \sigma_V^2 - 2\sigma_{LV}}$, where

$$\sigma_L^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} x_i\, x_j\, \rho_{ij}\, \sigma_i \sigma_j \text{ and}$$

$$\sigma_{LV} = \sum_{i=1}^{M} x_i\, \rho_{iV}\, \sigma_i\, \sigma_V .$$

Marginal Default Values – Lognormal Case

- Next, consider the effect of a marginal change in PV(losses) for a line of business. Since $D = f(L, V, \sigma)$, the total derivative of D with respect to L_i is:

$$\frac{\partial D}{\partial L_i} = \frac{\partial D}{\partial L}\frac{\partial L}{\partial L_i} + \frac{\partial D}{\partial V}\frac{\partial V}{\partial L_i} + \frac{\partial D}{\partial \sigma}\frac{\partial \sigma}{\partial L_i}$$

- The value of the default option for the company as a whole is:

$$\sum_{i=1}^{M} L_i\left(\frac{\partial D}{\partial L_i}\right) = \left(\frac{\partial D}{\partial L}\right)\sum_{i=1}^{M} L_i\left(\frac{\partial L}{\partial L_i}\right) + \left(\frac{\partial D}{\partial V}\right)\sum_{i=1}^{M} L_i\left(\frac{\partial V}{\partial L_i}\right) + \left(\frac{\partial D}{\partial \sigma}\right)\sum_{i=1}^{M} L_i\left(\frac{\partial \sigma}{\partial L_i}\right). \qquad (3)$$

Marginal Default Values – Lognormal Case

- Since $\frac{\partial L}{\partial L_i} = 1$ and $\frac{\partial V}{\partial L_i} = 1 + s$, it follows that

$$\sum_{i=1}^{M} L_i \left(\frac{\partial L}{\partial L_i} \right) = L \text{ and } \sum_{i=1}^{M} L_i \left(\frac{\partial V}{\partial L_i} \right) = V.$$

- Therefore, the RHS of (3) is

$$\frac{\partial D}{\partial L} L + \frac{\partial D}{\partial V} V + \left(\frac{\partial D}{\partial \sigma} \right) \sum_{i=1}^{M} L_i \left(\frac{\partial \sigma}{\partial L_i} \right) = D \;\; \textit{iff}$$

$$\left(\frac{\partial D}{\partial \sigma} \right) \sum_{i=1}^{M} L_i \left(\frac{\partial \sigma}{\partial L_i} \right) = 0.$$

Marginal Default Values – Lognormal Case

- (Very) tedious calculus results in the following expression:
  $$\frac{\partial \sigma}{\partial L_i} = \frac{1}{L}\frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})],$$ where σ_{iL} is the covariance of log losses in the i^{th} line of business with log losses on the insurance portfolio and σ_{iV} is the covariance of log losses on the i^{th} line of business with log asset values.
- Since $\sum_{i=1}^{M} x_i \sigma_{iL} = \sigma_L^2$ and $\sum_{i=1}^{M} x_i \sigma_{iV} = \sigma_{LV}$, the line-of-business default values add up to the default value for the company as a whole, since
  $$\sum_{i=1}^{M} L_i \left(\frac{\partial \sigma}{\partial L_i} \right) = \sum_{i=1}^{M} x_i \frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})] = 0!$$

Default Values & Surplus Requirements

- Next, MR derive formulas for computing marginal default values, which are needed for figuring out how to allocate surplus.

- The overall default-value-to-liability ratio is $d \equiv D / L$. Rearranging, $D = Ld$. Thus, the marginal default value is d_i, where

$$d_i \equiv \frac{\partial D}{\partial L_i} = d + L \frac{\partial d}{\partial x_i} \frac{\partial x_i}{\partial L_i} = d + L \frac{\partial d}{\partial x_i} \frac{1}{L} = \underbrace{d}_{\text{"scale" term}} + \underbrace{\partial d / \partial x_i}_{\text{"composition" term}} .$$

- The scale term captures the increase in overall default value due to an increase in PV(losses). The composition term captures the effect (positive or negative) of changing the firm's business mix.

Default Values & Surplus Requirements

- Next, we compute $\partial d / \partial x_i$. Note that $d \equiv D / L = f(L, V, \sigma) = f(V / L, \sigma) = f(1 + s, \sigma) = f(s, \sigma)$; thus,

$$\frac{\partial d}{\partial x_i} = \frac{\partial d}{\partial s}\frac{\partial s}{\partial x_i} + \frac{\partial d}{\partial \sigma}\frac{\partial \sigma}{\partial x_i}.$$

- We know $\frac{\partial \sigma}{\partial L_i} = \frac{\partial x_i}{\partial L_i}\frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})]$,

$$\therefore \frac{\partial \sigma}{\partial x_i} = \frac{\partial \sigma}{\partial L_i}\frac{\partial L_i}{\partial x_i} = \frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})].$$

Default Values & Surplus Requirements

- Next, we find $\frac{\partial s}{\partial x_i}$. Recall the product rule; i.e.,

  if $y = u(x)v(x)$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

- In the expression $s = S/L$, S corresponds to u and L^{-1} corresponds to v. Also note our earlier definition for L_i; i.e., $L_i = x_i L$. Therefore,

$$\frac{\partial s}{\partial x_i} = \frac{\partial S}{\partial L_i}\frac{\partial L_i}{\partial x_i}L^{-1} - S\frac{\partial L}{\partial L_i}\frac{\partial L_i}{\partial x_i}L^{-2}$$

$$= s_i(L)(L^{-1}) - S(1)(L)(L^{-2}) = s_i - s.$$

Default Values & Surplus Requirements

- Substituting these results into the original equation for d_i, we find that

$$d_i = d + \frac{\partial d}{\partial s}(s_i - s) + \left(\frac{\partial d}{\partial \sigma}\right)\left(\frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})]\right). \qquad (4)$$

  - Since the option "delta" ($\frac{\partial d}{\partial s}$) is negative, the higher the marginal surplus $(s_i - s)$, the lower the marginal default value, *cet. par.*
  - Vega ($\frac{\partial d}{\partial \sigma}$) is positive, so the higher σ_{iL} is, the higher the marginal default value, and the higher σ_{iV} is, the lower the marginal default value.

Default Values & Surplus Requirements

- Suppose $s_i = s$. Then

  $$d_i = d + \left(\frac{\partial d}{\partial \sigma}\right)\left(\frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})]\right),$$ which

  implies that marginal default values vary by line!

- The equal priority rule implies that if the company defaults on one policy, it defaults on all policies.

  - To ensure logical consistency with the equal priority rule, surplus allocation should equalize marginal default values; i.e., $d_i \equiv \dfrac{\partial D}{\partial L_i} = d$.

Default Values & Surplus Requirements

- Solving equation (4) for s_i results in the Myers-Read surplus allocation rule

$$s_i = s - \left(\frac{\partial d}{\partial s}\right)^{-1} \left(\frac{\partial d}{\partial \sigma}\right) \left(\frac{1}{\sigma}[(\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})]\right). \quad (6)$$

In (6), higher $\sigma_{iL} \Rightarrow$ higher marginal default value; also, higher $\sigma_{iV} \Rightarrow$ lower marginal default value.