Markets for Risk Management

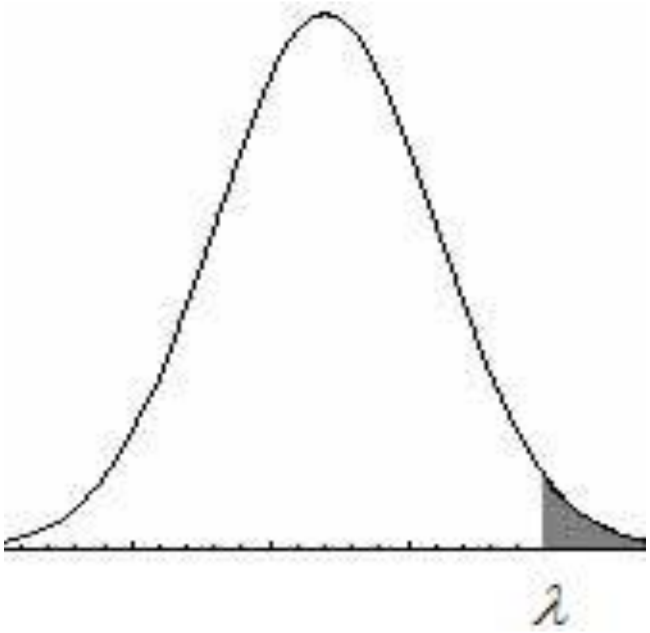
Financial Pricing Models (Part 1)

"Price Regulation in Property-Liability Insurance:
A Contingent Claims Approach"

Neil A. Doherty and James R. Garven

1986 *Journal of Finance*

Actuarial ("ruin theory") pricing model



- The level of surplus S that will produce an insolvency rate of p is $S = \lambda \sigma_{L_T}$.
- Total risk pool premium is $P_T = n\mu + \lambda \sigma_{L_T} = n\mu + \lambda \sqrt{n} \sigma$.
- $\therefore$ the premium per policyholder is $P_i = \underbrace{\mu}_{\text{pure premium}} + \underbrace{\frac{\lambda \sigma}{\sqrt{n}}}_{\text{risk loading}}$.

Valuation Relationships for a Property-Liability Insurer

- Beginning of period cash flow:

$$Y_0 = S_0 + P_0. \tag{1}$$

- End of period cash flow:

$$\tilde{Y}_1 = S_0 + P_0 + (S_0 + kP_0)\tilde{r}_i. \tag{2}$$

- In (2), *k* is the "funds generating coefficient"; measure of the average claim delay

Basic Valuation Relationships for a Property-Liability Insurer

- Y_1 is allocated to policyholders (H_1), government (T_1), and shareholders ($Y_1 - (H_1 + T_1)$).

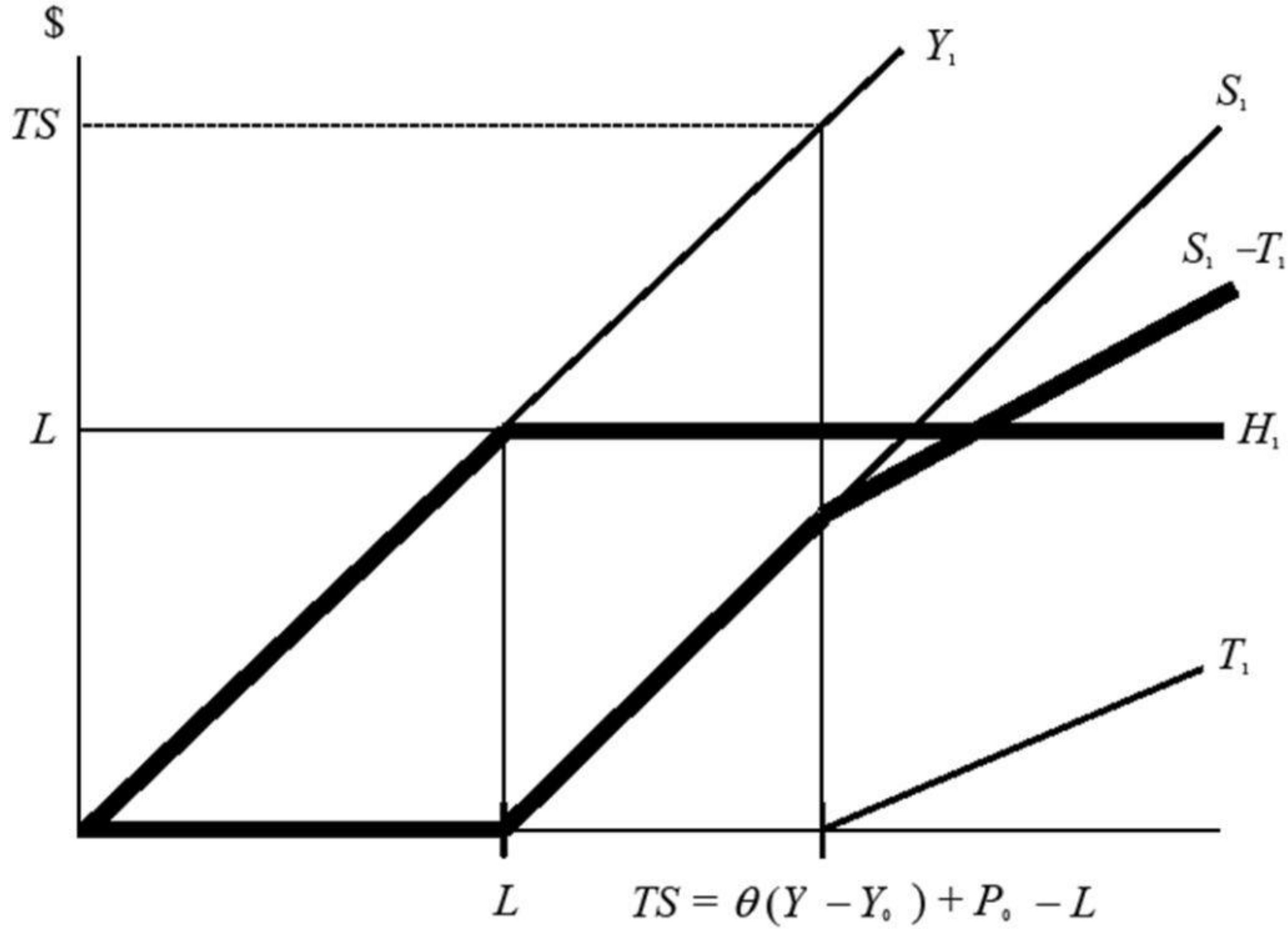
$$\tilde{H}_1 = \tilde{Y}_1 - Max[\tilde{Y}_1 - \tilde{L}, 0] = \tilde{L} - Max[\tilde{L} - \tilde{Y}_1, 0] \qquad (3a)$$

$$\tilde{T}_1 = \max[\tau(\theta(\tilde{Y}_1 - Y_0) + P_0 - \tilde{L}), 0], \qquad (4)$$

$$H_0 = V(\tilde{Y}_1) - C(\tilde{Y}_1; \tilde{L}) \qquad (5)$$

$$T_0 = \tau C[\theta(\tilde{Y}_1 - Y_0) + P_0; \tilde{L}], \qquad (6)$$

Basic Valuation Relationships for a Property-Liability Insurer



Basic Valuation Relationships for a Property-Liability Insurer

- Value of shareholders' claim V_e, is given by (7):

$$\begin{aligned} V_e &= V(\tilde{Y}_1) - [H_0 + T_0] \\ &= C[\tilde{Y}_1; \tilde{L}] - \tau C[\theta(\tilde{Y}_1 - Y_0) + P_0; \tilde{L}] \\ &= C_1 - \tau C_2. \end{aligned} \tag{7}$$

- Fair return $\Rightarrow$ NPV of investment in insurance = 0.

$$\begin{aligned} V_e &= C[\tilde{Y}_1(P_0^*); \tilde{L}] - \tau C[\theta(\tilde{Y}_1(P_0^*) - Y_0(P_0^*)) + P_0^*; \tilde{L}] \\ &= C_1^* - \tau C_2^* \\ &= S_0. \end{aligned} \tag{8}$$

Fair Return (CAPM)

$$V_e = R_f^{-1} \int_{-\infty}^{\infty} \tilde{Y}_e \hat{f}(\tilde{Y}_e)\, d\tilde{Y}_e$$

$$= R_f^{-1} \hat{E}(\tilde{Y}_e), \qquad (9)$$

where

$\tilde{Y}_e$ = random cash flow accruing to shareholders at the end of the period;
$\hat{f}(\tilde{Y}_e)$ = "risk-neutral" normal density function;
$\hat{E}(\tilde{Y}_e)$ = the certainty-equivalent expectation of $\tilde{Y}_e$
$= E(\tilde{Y}_e) - \lambda \operatorname{cov}(\tilde{Y}_e, \tilde{r}_m)$;
λ = the market price of risk
$= [E(\tilde{r}_m) - r_f]/\sigma_m^2$;
$\operatorname{cov}(\cdot)$ = the covariance operator.

Back to Fair Return (CAPM)

$$\hat{E}(\tilde{Y}_e) = S_0 + (1 - \theta\tau)\hat{E}(\tilde{r}_i)(S_0 + kP_0) + (1 - \tau)(P_0 - \hat{E}(\tilde{L})), \qquad (10)$$

where

$\hat{E}(\tilde{r}_i)$ = the certainty-equivalent expectation of rate of return on the insurer's investment portfolio
$= E(\tilde{r}_i) - \lambda \operatorname{cov}(\tilde{r}_i, \tilde{r}_m) = r_f$;
$\hat{E}(\tilde{L})$ = certainty-equivalent expectation of total claims costs
$= E(\tilde{L}) - \lambda \operatorname{cov}(\tilde{L}, \tilde{r}_m)$.

$$P_0 = \frac{E(\tilde{L})}{(1 - E(\tilde{r}_u))}, \qquad (11)$$

where

$$E(\tilde{r}_u) = [P_0 - E(\tilde{L})]/P_0$$

$$= -\frac{(1 - \theta\tau)}{(1 - \tau)} kr_f + (V_e/P_0)\frac{\theta\tau}{(1 - \tau)} r_f + \lambda \operatorname{cov}(\tilde{r}_u, \tilde{r}_m). \qquad (11a)$$

Some Special Cases

$$E(\tilde{r}_u) = -\frac{(1-\theta\tau)}{(1-\tau)} k r_f + (V_e/P_0)\frac{\theta\tau}{(1-\tau)} r_f + \lambda \operatorname{cov}(\tilde{r}_u, \tilde{r}_m). \qquad (11a)$$

- Case 1: No taxes or insolvency, zero-beta liabilities, $k = 1$; then $E(r_u) = -r_{f.}$
  - Implication: on average, insurer should lose money on underwriting.
- Case 2: Claim delays and correlated risks; then $E(r_u) = -kr_f + \beta_u[E(r_m) - r_f]$. claim
  - Insurer compensates the policyholder for delay, and there is a "risk load" for covariance risk.

Fair Return (CARA/Normal OPM)

- Value the call options (C_1 and C_2) described in equation (7) and solve for the implied fair premium (equation (8).
- First, solve for C_1 (Case 1: Joint Normality and Constant Absolute Risk Aversion).

$$C_1 = C[\tilde{Y}_1; \tilde{L}]$$

$$= R_f^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max[(\tilde{Y}_1 - \tilde{L}), 0] \hat{f}(\tilde{Y}_1, \tilde{L})\, d\tilde{Y}_1\, d\tilde{L}, \qquad (12)$$

where $\hat{f}(\tilde{Y}_1, \tilde{L})$ is the bivariate risk-neutral density function governing the realization of the normal variates $\tilde{Y}_1$ and $\tilde{L}$.

Fair Return (CARA/Normal OPM)

Next, we simplify equation (12) by defining a normal variate $\tilde{X} = \tilde{Y}_1 - \tilde{L}$, with certainty-equivalent expectation $\hat{E}(\tilde{X}) = \hat{E}(\tilde{Y}_1) - \hat{E}(\tilde{L}) = S_0 + (S_0 + kP_0)r_f + P_0 - \hat{E}(\tilde{L})$, and variance $\sigma_x^2 = (S_0 + kP_0)^2\sigma_i^2 + \sigma_L^2 - 2(S_0 + kP_0)\mathrm{cov}(\tilde{L}, \tilde{r}_i)$.

$$C_1 = R_f^{-1} \int_0^\infty \tilde{X}\hat{f}(\tilde{X})\, d\tilde{X}. \tag{13}$$

Since $\tilde{X}$ is normally distributed, equation (13) may be rewritten in terms of the standard normal variate $\tilde{z} = (\tilde{X} - \hat{E}(\tilde{X}))/\sigma_x$; hence,

$$C_1 = R_f^{-1}(2\pi)^{-1/2} \int_{-\hat{E}(\tilde{X})/\sigma_x}^{\infty} [\hat{E}(\tilde{X}) + \sigma_x\tilde{z}]e^{-\tilde{z}^2/2}\, d\tilde{z}. \tag{14}$$

The solution for equation (14) is (15):

$$C_1 = R_f^{-1}(\hat{E}(\tilde{X})N[\hat{E}(\tilde{X})/\sigma_x] + \sigma_x n[\hat{E}(\tilde{X})/\sigma_x]), \tag{15}$$

Partial Moment Mathematics

- Winkler, Roodman, and Britney (1972 *Management Science*) show that the n^th partial moment of a normally distributed random variable is written

$$E_{-\infty}^{z}(X^n) = -\sigma^2 z^{n-1} f(z) + (n-1)\sigma^2 E_{-\infty}^{z}(X^{n-2}) + \mu E_{-\infty}^{z}(X^{n-1}).$$

Suppose $n = 1$. Then $E_{-\infty}^{z}(X) = -\sigma^2 f(z) + \mu F(z)$. Also, $E_{z}^{\infty}(X) = \sigma^2 f(z) + \mu F(z)$. Applying this result to (14),

$$R_f C_1 = \hat{E}(X) N[\hat{E}(X)/\sigma_X] + \sigma_X \int_{\frac{-\hat{E}(X)}{\sigma_X}}^{\infty} z n(z) dz$$

$$= \hat{E}(X) N[\hat{E}(X)/\sigma_X] + \sigma_X \left(E(z) N[\hat{E}(X)/\sigma_X] + \sigma_z^2 n[\hat{E}(X)/\sigma_X] \right)$$

$$= \hat{E}(X) N[\hat{E}(X)/\sigma_X] + \sigma_X \left(0 \times N[\hat{E}(X)/\sigma_X] + 1 \times n[\hat{E}(X)/\sigma_X] \right)$$

$$= \hat{E}(X) N[\hat{E}(X)/\sigma_X] + \sigma_X n[\hat{E}(X)/\sigma_X].$$

Fair Return (CARA/Normal OPM)

The value of the second call option, C_2, may be written as the discounted certainty-equivalent expectation of the insurer's terminal taxable income, viz.,

$$C_2 = C[\theta(\tilde{Y}_1 - Y_0) + P_0; \tilde{L}]$$

$$= R_f^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max[\theta(\tilde{Y}_1 - Y_0) + P_0 - \tilde{L}, 0] \hat{f}(\tilde{Y}_1, \tilde{L}) \, d\tilde{Y}_1 \, d\tilde{L}. \quad (16)$$

Next, we simplify equation (16) by defining a normal variate $\tilde{W} = \theta(\tilde{Y}_1 - Y_0) + P_0 - \tilde{L}$, with certainty-equivalent expectation $\hat{E}(\tilde{W}) = \theta(S_0 + kP_0)r_f + P_0 - \hat{E}(\tilde{L})$, and variance $\sigma_w^2 = (S_0 + kP_0)^2\theta^2\sigma_i^2 + \sigma_L^2 - 2(S_0 + kP_0)\theta \operatorname{cov}(\tilde{L}, \tilde{r}_i)$. This transformation allows us to rewrite our option value as the solution to

$$C_2 = R_f^{-1} \int_0^{\infty} \tilde{W}\hat{f}(\tilde{W}) \, d\tilde{W}. \quad (17)$$

Fair Return (CARA/Normal OPM)

$$C_2 = R_f^{-1}(\hat{E}(\tilde{W})N[\hat{E}(\tilde{W})/\sigma_w] + \sigma_w n[\hat{E}(\tilde{W})/\sigma_w]), \quad (18)$$

where

$N[\hat{E}(\tilde{W})/\sigma_w]$ = the standard normal distribution evaluated at $\hat{E}(\tilde{W})/\sigma_w$;
$n[\hat{E}(\tilde{W})/\sigma_w]$ = the standard normal density evaluated at $\hat{E}(\tilde{W})/\sigma_w$.

Substituting the right-hand sides of equations (15) and (18) into equation (7), we obtain an analytic expression for the market value of equity:

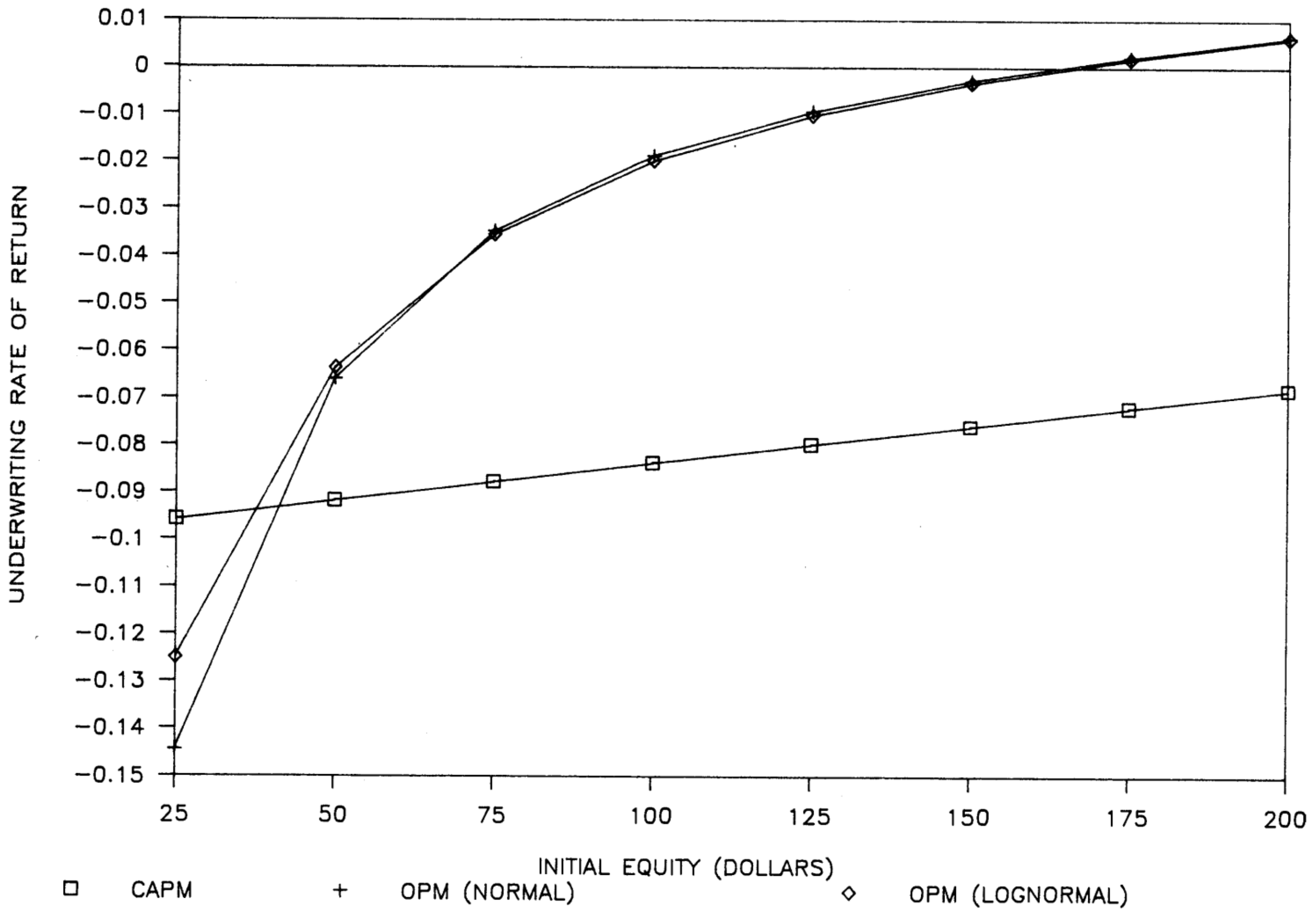
$$V_e = R_f^{-1}(\hat{E}(\tilde{X})N[\hat{E}(\tilde{X})/\sigma_x] - \tau\hat{E}(\tilde{W})N[\hat{E}(\tilde{W})/\sigma_w] + \sigma_x n[\hat{E}(\tilde{X})/\sigma_x] - \tau\sigma_w n[\hat{E}(\tilde{W})/\sigma_w]). \quad (19)$$

An implicit solution for the value of P_0 that satisfies the fair return criterion implied by equation (8) may be obtained by employing an appropriate algorithm.

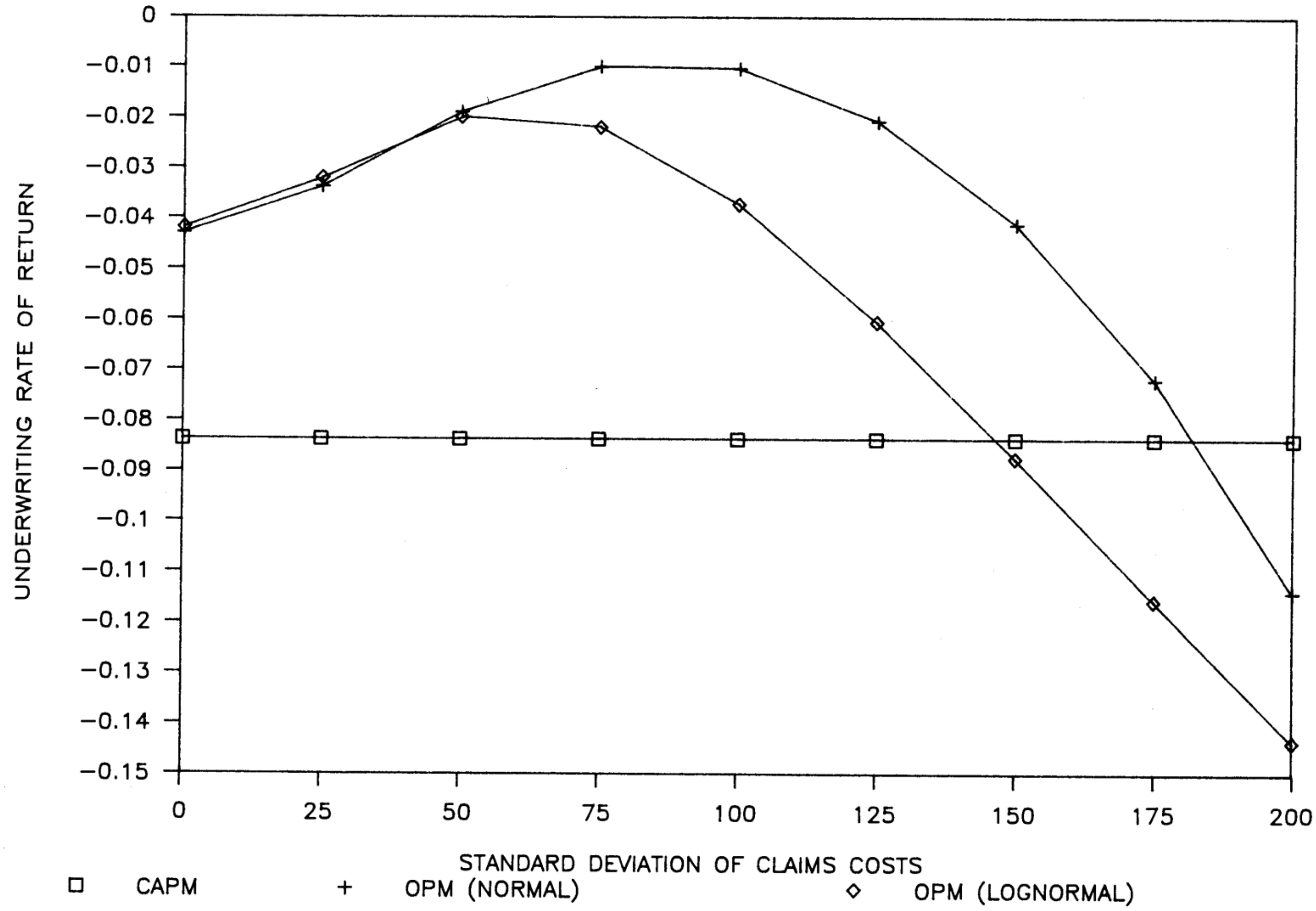
Model Parameterization

Initial Equity (S_o)	100.00
Funds-Generating Coefficient (k)	1.00
Standard Deviation of Investment Returns (σ_i)	0.20
Expected Claims Costs ($E(\tilde{L})$)	200.00
Standard Deviation of Claims Costs (σ_L)	50.00
Correlation Between Investment Returns/Claims Costs (ρ_{iL})	0.00
Riskless Rate of Interest (r_f)	0.07
Statutory Tax Rate (τ)	0.46
Tax-Adjustment Parameter (θ)	0.50
Beta of Investment Portfolio (β_i)	0.338
Expected Return on the Market ($E(\tilde{r}_m)$)	0.15
Standard Deviation of Market Return (σ_m)	0.224

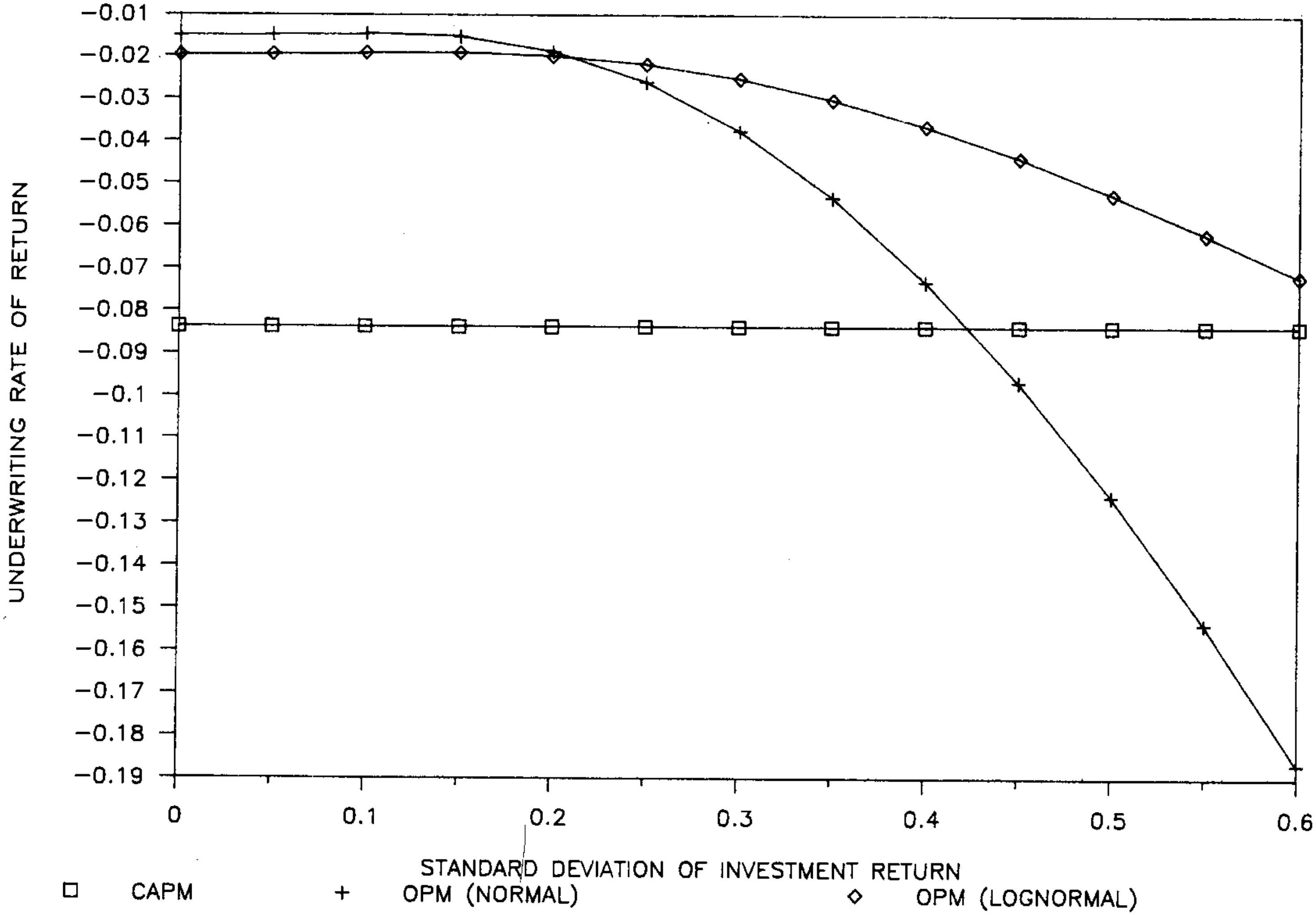
Vary Level of Initial Equity



Vary Standard Deviation of Claims Costs



Vary Standard Deviation of Investment Returns



Numerical Comparisons: Default Risk

CAPM: If $\tau = 0$, then the "fair" price for insurance $P_0 = E(L)/(1-E(r_u))$, where $E(r_u) = -kr_f + \beta_u[E(r_m) - r_f]$. Let $E(L) = k = 1$, $\beta_u = 0$, $r_f = 7\%$. Then $P_0 = 1/1.07 = \$.9345$.

OPM: Same parameterization as for CAPM, only solve equation (19) subject to the fair return criterion given by equation (8), for surplus values ranging from \$3 to \$.10. Also assume:

(1) standard deviation of claims costs (σ_L) = \$.40,

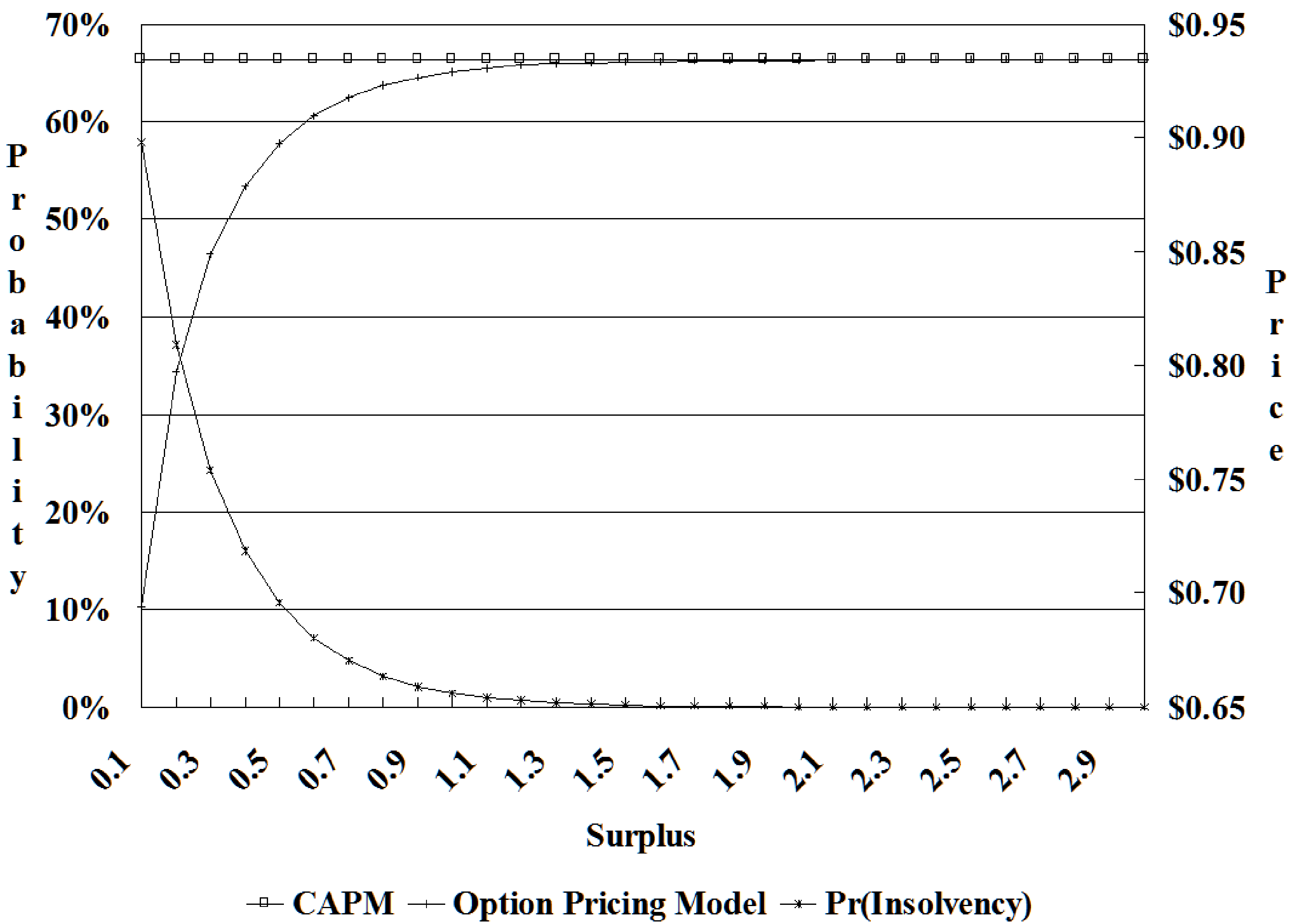
(2) market risk premium ($E(r_m)-r_f$) = 8 percent,

(3) standard deviation of market return (σ_m) = 20 percent,

(4) correlation between investment returns and claims costs (ρ_{iL}) = 0, and

(5) beta of insurer's investments (β_i) = 1.

Numerical Comparisons: Default Risk



Numerical Comparisons: Tax Effects